

# A MALL Geometry of Interaction Based on Indexed Linear Logic

Masahiro HAMANO

Institute of Information Science, Academia Sinica, Taiwan

hamano@jaist.ac.jp

April 11, 2017

## Abstract

We construct a geometry of interaction (GoI; dynamic modeling of Gentzen-style cut elimination) for multiplicative-additive linear logic (MALL) by employing the Bucciarelli–Ehrhard MALL( $I$ ) of indexed linear logic to handle the additives. Our construction is an extension to the additives of the Haghverdi–Scott categorical formulation (a multiplicative GoI situation in a traced monoidal category) for Girard’s original GoI  $I$ . The indexes are shown to serve not only in their original denotational level, but also at a finer grained dynamic level so that the peculiarities of additive cut elimination such as superposition, erasure of subproofs, and additive (co-) contraction can be handled with the explicit use of indexes. Proofs are interpreted as indexed subsets in the category Rel, but without the explicit relational composition; instead, execution formulas are run pointwise on the interpretation at each index, w.r.t symmetries of cuts, in a traced monoidal category with a reflexive object and a zero morphism. The indexes diminish overall when an execution formula is run, corresponding to the additive cut-elimination procedure (erasure), and allowing recovery of the relational composition. The main theorem is the invariance of cut elimination via convergence of the execution formulas on the denotations of (cut-free) proofs.

## 1 Introduction

The indexed multiplicative additive linear logic MALL( $I$ ), introduced by Bucciarelli–Ehrhard [Bucciarelli and Ehrhard ’00], is a conservative extension of Girard’s MALL in which all formulas and proofs come equipped with sets of indexes. The usual MALL is stipulated to be the restriction to the empty set. The status of the indexed syntactical system is noteworthy as it stems from the denotational semantics of Rel, a simple, yet pivotal categorical model of MALL. With the enabling of an explicit notion of location in linear proof theory, the indexes can enumerate the locations of formulas and proofs, corresponding to denotational interpretations in MALL. The notion of location becomes a requirement for the additives, although it is redundant for the multiplicatives, for which the singleton  $\{*\}$  suffices. To work with *parallelism*, which the additives bring intrinsically, different locations need to be handled rather than the sole location  $*$ . In the context of parallelism, superpositions are known to typically arise under the syntactic additive  $\&$ -rule. Indexes allow one to deal with superpositions by identifying multiple occurrences of formulas in the different indexes and enlarging (or restricting) the indexes.

The original motivation for indexed logic was to provide a bridge between a truth-valued semantics (for provability) for MALL( $I$ ) and the denotational semantics of (nonindexed) MALL. By means of this bridge, Bucciarelli–Ehrhard obtained a new kind of denotational completeness theorem in [Bucciarelli and Ehrhard ’00] for MALL and later extended it to the exponentials in [Bucciarelli and Ehrhard ’01].

This paper investigates indexed MALL from the perspective of a dynamic semantics for cut elimination, a topic that—to the best of our knowledge—has remained untouched aside from

the precursory work of Duchesne [Duchesne '09] since the original work of Bucciarelli–Ehrhard [Bucciarelli and Ehrhard '00]. The dynamic semantics is known as the Girard project of Geometry of Interaction (GoI), whereby cut-elimination is modelled, using operator algebras [Girard '89] and more generally traced monoidal categories [Joyal *et al.* '96]. The project was successful [Girard '89, Haghverdi Scott '06] in MLL with the exponentials, and inspired a new model of computation for  $\beta$  reduction of  $\lambda$ -calculus [Danos and Regnier '95]. This paper aims to initiate an exploration how to combine the two notion of *location*, which the indexed logic brings, and of *dynamics*, which GoI brings to cut-elimination. The combination is important in understanding additive cut-elimination. For this goal, we employ the indexes to construct a GoI model for (non-indexed) MALL. We combine the Haghverdi–Scott categorical GoI situation [Haghverdi Scott '06] with the indexes in such a way that that the original MLL GoI situation represents a collapse to the singleton index  $\{*\}$ . The dynamism of cut-elimination is captured by a feedback mechanism determined by traces of morphisms interpreting proofs. We furthermore augment the situation with two kind of actions, identical and zero, over the symmetries interpreting the cut-rule. These two actions provide representations of matches and of mismatches among locations, and come into play during a Gentzen style cut-elimination procedure, in which one encounters noncommunication of individual proofs, as a result of the additive parallelism. Crucial instances of GoI situation such as  $\text{Rel}_+$  and  $\text{Hilb}_2$  [Haghverdi Scott '06] are directly lifted to our framework, the latter of which is the operator algebraic origin of the Girard project.

We study Girard’s execution formula [Girard '89] in the general categorical setting of a traced symmetric monoidal category. The execution formula accommodates indexes, and faithfully simulates MALL cut-elimination by a hybrid method to relate the indexed syntax to the relational semantics. Each location in the corresponding relation of a proof is first assigned an endomorphism on a reflexive object  $U$ . The cut-rule before execution is interpreted as a tensor product of two premise morphisms, more loosely than their composition. This interpretation allows extraction of the dynamical meaning of the cut, which the usual categorical composition makes hidden by virtue of its static approach. In the loose interpretation, there remain redundant indexes when interpreting rules, however, they are shown to disappear, while running Execution formula in terms of categorical trace structure. The disappearance of indexes is modelled by zero morphisms, which exist in crucial instances of traced monoidal category for GoI. Proof-theoretically, the zero morphisms provide to interpret discarding subproofs peculiar to additive cut-elimination, and Index-theoretically, they provide to interpret mismatching among locations. In traced monoidal category, the zero morphisms are formulated to act partially on symmetries for cut-formulas, and also to act partially on retractions and co-retractions of the reflexive objects. The latter action arises via tracing which takes feedback into account. The zero-convergence is proved that execution formulas converge to zero when two proofs interact with mismatching. Thus the execution formulas terminate to denotational interpretations of proofs with diminution of indexes in order to recover the relational composition. This is typically realized by properly coupling indexes to trace axioms, especially “generalized yanking”, which directly designates that traces are primitive enough to retrieve the categorical composition in a monoidal category, as well as “dinaturality” for interaction of bidirectional flow of morphisms.

We prove two main result: (i) (Invariance of the execution formula during MALL normalization): Execution formula in our dynamic categorical modelling is shown to converge to the denotational interpretation of proofs in the static categorical model. This characterizes the normalization for proofs by categorical invariant. (ii) (Diminution of indexes while running execution formula): Execution may converge to 0, making the redundant indexes disappear. Part (i) is seen as a pointwise collection of invariants, as previously established for the multiplicatives [Girard '89, Haghverdi Scott '06]. Part (ii) is peculiar to the additives; –proof-theoretically– it reflects erasure as well as additive (co) contraction and superposition, in cut-elimination and–category-theoretically–it ensures that our categorical ingredient of execution formula is finer grained enough to retrieve a static monoidal category as well as relational category handling indexes.

## 1.1 Prologue: Indexes and additive cut elimination

Consider a sequence  $\pi_1 \triangleright \pi_2 \triangleright \pi_3$  of cut eliminations for proofs in the additive fragment of MALL. In our sequent notation, pairwise cut formulas, if present, are stored inside a stack  $[\cdot]$  in a sequent. The first reduction, intrinsic to the additives, eliminates a  $\&$  in a cut, whereby the subproof  $\text{ax}_2$  is pruned. The second reduction eliminates a redundant cut against an axiom:

$$\frac{\frac{\overline{\vdash A^\perp, A} \text{ ax}_1 \quad \overline{\vdash A^\perp, A} \text{ ax}_2}{\vdash A^\perp, A \& A} \& \quad \frac{\overline{\vdash A^\perp, A} \text{ ax}_3}{\vdash A^\perp \oplus A^\perp, A} \oplus_1}{\vdash [A \& A, A^\perp \oplus A^\perp] A^\perp, A} \text{ cut} \quad \triangleright \quad \frac{\overline{\vdash A^\perp, A} \text{ ax}_1 \quad \overline{\vdash A^\perp, A} \text{ ax}_3}{\vdash [A, A^\perp] A^\perp, A} \text{ cut} \\ \triangleright \quad \overline{\vdash A^\perp A} \text{ ax}_1$$

### Step 1 (Interpretation $|\pi|$ in Rel with unperformed cuts and indexes for additives)

We begin by interpreting proofs in Rel but without relational composition. For this, the cut rule is interpreted in parallel to the tensor rule. This interpretation is consistent with the syntactic convention, starting from Girard's GoI I, which puts the cut formulas into a stack.

For simplicity and in accordance with the fact that the multiplicative dual elements 1 and  $\perp$  are interpreted in Rel as the singleton set, we take  $A = 1$  and, dually,  $A^\perp = \perp$ , so that  $|1| = |\perp|$  is the singleton, whose unique element is denoted  $*$  or  $\bar{*}$  (obviously,  $*$  or  $\bar{*}$ ), depending on whether it comes from  $|A|$  or  $|A^\perp|$ , respectively.

An axiom is interpreted in Rel by the diagonal, so that  $|\text{ax}_i| = \{(\bar{*}, *)\} \subseteq |A^\perp| \times |A|$ . The proof  $\pi_2$  is interpreted as  $|\text{cut}(\text{ax}_1, \text{ax}_2)| = \{(\bar{*}, *, \bar{*}, *)\} \subseteq |A^\perp| \times [|A| \times |A^\perp|] \times |A|$ , in which the pair  $(*, \bar{*})$  in the cut slot from  $[|A| \times |A^\perp|]$  remains explicit, rather than being hidden by relational composition through  $*$  or  $\bar{*}$ . Note that both interpretations  $|\pi_2|$  and  $|\pi_3|$  are singletons. More generally, it is straightforward to see that any proof in the multiplicatives can be interpreted by a singleton whenever literals are interpreted by singletons. However, this is not the case for the additives. When interpreting the  $\pi_1$  with additive rules, singletons proves insufficient, and this is where the indexes become necessary: The left and right premises of  $\pi_1$  are interpreted respectively by  $|\&(\text{ax}_1, \text{ax}_2)| = \{(\bar{*}, (1, *)), (\bar{*}, (2, *))\} \subseteq |A^\perp| \times (|A| + |A|)$  and  $|\oplus_1(\text{ax}_3)| = \{((1, \bar{*}), *)\} \subseteq (|A^\perp| + |A^\perp|) \times |A|$ . A set of indexes  $J = \{1, 2\}$  is employed to describe these two interpretations: the first yields  $\delta \in |\&(\text{ax}_1, \text{ax}_2)|^J$  so that  $\delta_1 = (\bar{*}, (1, *))$  and  $\delta_2 = (\bar{*}, (2, *))$ , and the second yields  $\tau \in |\oplus_1(\text{ax}_3)|^J$ , so  $\tau_1 = \tau_2 = ((1, \bar{*}), *)$ .

Then  $\pi_1$  is interpreted by  $\nu$ :

$$\nu := \delta \times \tau \in |\text{cut}(\&(\text{ax}_1, \text{ax}_2), \oplus_1(\text{ax}_3))|^J \subseteq (|A^\perp| \times [(|A| + |A|) \times (|A^\perp| + |A^\perp|)] \times |A|)^J,$$

and therefore  $(\delta \times \tau)_1 = \delta_1 \times \tau_1 = (\bar{*}, (1, *), (1, \bar{*}), *)$  and  $(\delta \times \tau)_2 = \delta_2 \times \tau_2 = (\bar{*}, (2, *), (1, \bar{*}), *)$ .

### Step 2 ( $\text{Ex}_J(\sigma, \nu)$ for $\nu \in |\pi|^J$ : Executing cuts using trace structures)

In addition to step 1, our GoI interpretation runs an execution formula for  $|\pi_i|$  to perform cut elimination against the unperformed cut formulas, syntactically in the stack and semantically in the noncompositional interpretation.

Each point in  $|\pi|$  is interpreted as an endomorphism on a certain tensor folding of a *reflexive object*  $U$  in a traced monoidal category  $\mathcal{C}$  with a zero morphism on  $U$ . The object  $U$  uniformly interprets each element in the interpretation of the conclusion of  $\pi$ ; e.g., in  $|\pi_1|$ ,  $U$  yields the elements  $\bar{*}$ ,  $*$ ,  $(1, \bar{*})$ ,  $(1, *)$ , and  $(2, *)$ . In the following, these points are identified with their interpretation  $U$ .

As the most primitive case, e.g., for  $|\pi_3|$ , each diagonal point  $(\bar{*}, *)$  is interpreted as a *symmetry* of  $\mathcal{C}$ :

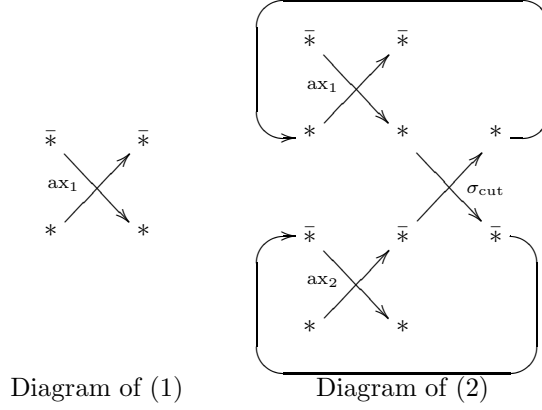
$$s_{\bar{*},*} : U_{\bar{*}} \otimes U_* \longrightarrow U_{\bar{*}} \otimes U_* \quad (1)$$

The unique point of  $|\pi_2|$  is interpreted by the endomorphism  $\text{Ex}(\sigma_{\text{cut}}, |\pi_2|)$  on  $U_{\bar{*}} \otimes U_*$ , in which  $\sigma_{\text{cut}}$ , as the interpretation of the cut, is the symmetry  $s_{*,\bar{*}}$  acting on the cut formulas:

$$\text{Ex}(\sigma_{\text{cut}}, |\pi_2|) = \text{Tr}_{\bar{*} \otimes *, \bar{*} \otimes *}^{* \otimes \bar{*}} ((\bar{*} \otimes \sigma_{\text{cut}} \otimes *) \circ (s_{\bar{*},*} \otimes s_{*,*})). \quad (2)$$

This is equal to (1) in  $\mathcal{C}$  by the trace axioms. The adjacent diagrams illustrate (1) and for (2), where the equality is found in the diagram for (2) by following arrows with respect to both composition and feedback.

The GoI interpretation of  $|\pi_1|$  is that for the indexed  $\nu \in |\pi_1|^J$ , which is defined point-wise (for  $j \in J = \{1, 2\}$ ) at  $\nu_1$  and  $\nu_2$ , in which  $\sigma_{\nu_1}$  and  $\sigma_{\nu_2}$  are stipulated respectively by  $\sigma_{\text{cut}}$  and 0, where  $\sigma_{\text{cut}}$  is a symmetry  $s_{(1,*),(1,\bar{*})}$  for the cut formulas while 0 is the zero morphism acting on a symmetry  $s_{(2,*),(1,\bar{*})}$ :



$$\text{Ex}(\sigma_{\nu_1}, \nu_1) = \text{Tr}_{\bar{*} \otimes *, * \otimes \bar{*}}^{(1,*) \otimes (1,\bar{*})} ((\bar{*} \otimes \sigma_{\text{cut}} \otimes *) \circ (s_{\bar{*},(1,*)} \otimes s_{(1,\bar{*}),*})), \quad (3)$$

in which  $\sigma_{\text{cut}}$  arises because of the matching  $(1, *) = (1, \bar{*})$ .

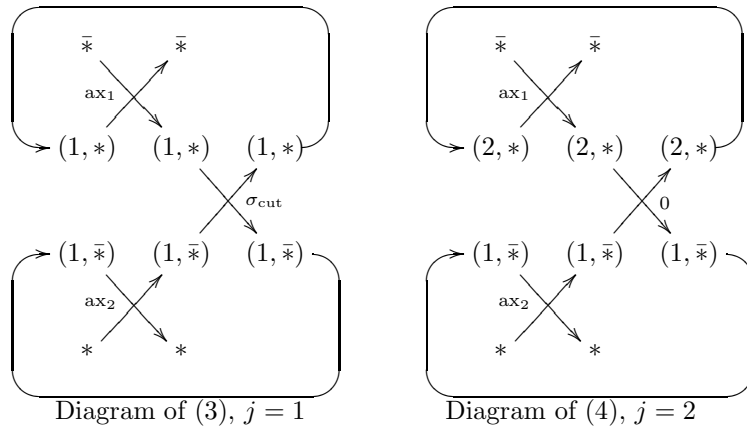
$$\text{Ex}(\sigma_{\nu_2}, \nu_2) = \text{Tr}_{\bar{*} \otimes *, * \otimes \bar{*}}^{(2,*) \otimes (1,\bar{*})} ((\bar{*} \otimes 0 \otimes *) \circ (s_{\bar{*},(2,a)} \otimes s_{(1,\bar{*}),*})) \quad (4)$$

in which 0 arises because of the mismatch  $(2, *) \neq (1, \bar{*})$ .

Here (3) is equivalent to (2), while (4) reduces to 0 in  $\mathcal{C}$  by virtue of the trace axioms for zero morphisms. The next two diagrams, for (3) and (4), illustrate that (4) yields a zero morphism because following any arrow results in passing through 0.

At  $j = 2$ ,

$\text{Ex}(\sigma_{\nu_2}, \nu_2) = 0$ , so we delete the index 2, reducing  $J$  into the singleton  $\{1\}$ . For index 1,  $\text{Ex}(\sigma_{\nu_1}, \nu_1)$  is identical to the symmetry (1) of  $|\pi_3|$  and, hence, to the denotational interpretation.



The rest of this paper is organized as follows: Section 2 introduces a syntax  $\text{MALL}^{[c]}(I)$  for indexed MALL with a cut list as well as its relational counterpart  $\text{Rel}^{[c]}$ . A fundamental lemma is proved, which connects a provable  $\text{MALL}^{[c]}(I)$  sequent to an indexed subset of the interpretation of a MALL proof with cuts. In Section 3, MALL proof reduction for cut elimination is lifted to  $\text{MALL}(I)$  proof transformation with diminishing indexes. Section 4 concerns our MALL GoI interpretation by means of the indexed system in a traced symmetric monoidal category with zero action. Execution formulas are run indexwise, and the main theorem is proved.

## 2 MALL( $I$ ) with cut list and relational semantics

### 2.1 MALL( $I$ ) with cut list

**(Inference rules of MALL<sup>[c]</sup> with cut formulas)**

We accommodate a stack to record cut formulas into the syntax of the multiplicative-additive linear logic MALL. To stress this, the system is written as MALL<sup>[c]</sup>. To accommodate in the additive fragment, one has to work with superpositions that arise inside the stack as well as in the conclusion (outside the stack).

A MALL sequent  $\vdash [\Delta], \Gamma$  with a cut list is a set  $\Gamma$  of formula occurrences together with pairwise-dual formulas occurrences  $\Delta$  inside the brackets. Each dual pair in  $\Delta$  is written  $A, A^\perp$ , in that order. Sequents are proved using the following rules:

$$\begin{array}{c}
\frac{}{\vdash A, A^\perp} \text{ ax} \quad \frac{\vdash [\Delta_1], \Gamma_1, A \quad \vdash [\Delta_2], \Gamma_2, B}{\vdash [\Delta_1, \Delta_2], \Gamma_1, \Gamma_2, A \otimes B} \otimes \quad \frac{\vdash [\Delta], \Gamma, A, B}{\vdash [\Delta], \Gamma, A \wp B} \wp \\
\\
\frac{\vdash [\Delta_1], \Gamma_1, A \quad \vdash [\Delta_2], A^\perp, \Gamma_2}{\vdash [\Delta_1, \Delta_2, A, A^\perp], \Gamma_1, \Gamma_2} \text{ cut} \\
\\
\frac{\vdash [\Delta_1, \Delta], \Gamma, A \quad \vdash [\Delta_2, \Delta], \Gamma, B}{\vdash [\Delta_1, \Delta_2, \Delta], \Gamma, A \& B} \& \quad \frac{\vdash [\Delta], \Gamma, A}{\vdash [\Delta], \Gamma, A \oplus B} \oplus_1 \quad \frac{\vdash [\Delta], \Gamma, B}{\vdash [\Delta], \Gamma, A \oplus B} \oplus_2
\end{array}$$

**Note:** In the  $\&$ -rule, not only is  $\Gamma$  superposed in the conclusion, but so is  $\Delta$  in the stack. The superposition among cut formulas inside the stack causes the well-known additive (co-) contraction that arises in MALL cut elimination. The exchange rule is eliminated under the assumption that formula occurrences are implicitly tracked by premises and conclusion of a rule.

We extend the above accommodation of cut lists to the Bucciarelli–Ehrhard [Bucciarelli and Ehrhard '00] indexed system MALL( $I$ ). To stress this, the system is written as MALL<sup>[c]</sup>( $I$ ). The extension stipulates that a set of indexes is consistently associated with each formula (including cut formulas) and sequent (including cut lists).

We fix an index set  $I$ , once and for all. Each formula  $A$  of MALL( $I$ ) is associated with a set  $d(A) \subseteq I$ , called the *domain* of  $A$ .

**(MALL( $I$ ) formulas and domains)**

Formulas in the domain  $J$  are defined by the following grammar: For any  $J, K, L \subseteq I$  such that  $K \cap L = \emptyset$ ,

$$A_J ::= \mathbf{1}_J \mid \perp_J \mid \mathbf{0}_\emptyset \mid T_\emptyset \mid A_J \otimes A_J \mid A_J \wp A_J \mid A_K \oplus A_L \mid A_K \& A_L.$$

For any MALL( $I$ ) formula  $A$  with  $d(A) = J$ , its negation  $A^\perp$  with  $d(A^\perp) = J$  is defined using the De Morgan duality for the MALL formula.

**(Restriction)**

For a MALL( $I$ ) formula  $A$  with  $d(A) = J$  and  $K \subseteq J$ , the restriction  $A \upharpoonright_K$  of  $A$  by  $K$  is defined to be a MALL( $I$ ) formula in the domain  $J \cap K$  as follows:

$$\begin{array}{l}
\mathbf{0}_\emptyset \upharpoonright_K = \mathbf{0}_\emptyset \text{ and } T_\emptyset \upharpoonright_K = T_\emptyset. \quad \perp_J \upharpoonright_K = \perp_{J \cap K} \text{ and } \mathbf{1}_J \upharpoonright_K = \mathbf{1}_{J \cap K} \quad (P \otimes Q) \upharpoonright_K = P \upharpoonright_K \otimes Q \upharpoonright_K \\
(P \wp Q) \upharpoonright_K = P \upharpoonright_K \wp Q \upharpoonright_K \quad (P \oplus Q) \upharpoonright_K = P \upharpoonright_K \oplus Q \upharpoonright_K \quad (P \& Q) \upharpoonright_K = P \upharpoonright_K \& Q \upharpoonright_K
\end{array}$$

It trivially follows that  $(A^\perp) \upharpoonright_K = (A \upharpoonright_K)^\perp$ . If  $\Gamma$  is a sequence of MALL( $I$ ) formulas  $A_1, \dots, A_n$  of domains  $J$ , we define  $\Gamma \upharpoonright_K = A_1 \upharpoonright_K, \dots, A_n \upharpoonright_K$ .

**(Inference rules of MALL<sup>[c]</sup>( $I$ ) with cut lists)**

We augment MALL<sup>[c]</sup> with indexes. This makes it possible to accommodate a stack to hold cut formulas as in the original MALL( $I$ ) [Bucciarelli and Ehrhard '00]. Although this is straightforward for the multiplicative fragment, careful treatment is required for the listing of cut formulas in the additive fragment. For a sequence  $\Xi$  of formulas, we write  $d(\Xi) = J$  (resp.  $d(\Xi) \subseteq J$ ) when  $d(A) = J$  (resp.  $d(A) \subseteq J$ ) for any formula  $A$  in  $\Xi$ .

Each sequent is of the form  $\vdash_J [\Delta], \Gamma$ , in which  $d(\Gamma) = I$  and  $d(\Delta) \subseteq J$  with  $d(A) = d(A^\perp) \subseteq J$  for any pairwise-dual formulas  $A$  and  $A^\perp$  in  $\Delta$  within the stack.

**Note:** The uniformity requirement that all formulas in  $\Gamma$  have the same domain  $I$  does not apply to the stack  $\Delta$  of the cut formulas. Formulas from different cuts have varied domains contained in  $J$ .

**Axioms and cut:**

$$\vdash_J \mathbf{1}_J \quad \vdash_\emptyset \Gamma, T_\emptyset \quad \frac{\vdash_J [\Delta_1], \Gamma_1, A \quad \vdash_J [\Delta_2], A^\perp, \Gamma_2}{\vdash_J [\Delta_1, \Delta_2, A, A^\perp], \Gamma_1, \Gamma_2} \text{ cut} \quad \text{Note that } d(A) = d(A^\perp) = J \text{ for cut formulas } A \text{ and } A^\perp.$$

**Multiplicative rules:**

$$\frac{\vdash_J [\Delta], \Gamma}{\vdash_J [\Delta], \Gamma, \perp_J} \perp_J \quad \frac{\vdash_J [\Delta_1], \Gamma_1, A \quad \vdash_J [\Delta_2], \Gamma_2, B}{\vdash_J [\Delta_1, \Delta_2], \Gamma_1, \Gamma_2, A \otimes B} \otimes \quad \frac{\vdash_J [\Delta], \Gamma, A, B}{\vdash_J [\Delta], \Gamma, A \wp B} \wp$$

**Additive rules:**

$$\frac{\vdash_{J_1} [\Delta_1, \Sigma \upharpoonright_{J_1}], \Gamma \upharpoonright_{J_1}, A_1 \quad \vdash_{J_2} [\Delta_2, \Sigma \upharpoonright_{J_2}], \Gamma \upharpoonright_{J_2}, A_2}{\vdash_{J_1+J_2} [\Delta_1, \Delta_2, \Sigma], \Gamma, A_1 \& A_2} \&$$

Note that the superposed context  $\Gamma$  in the concluding rule encompasses the *whole* domain  $J = J_1 + J_2$ , while the superposed context  $\Sigma$  in the stack has a domain *contained in*  $J$ .

$$\frac{\vdash_J [\Delta], \Gamma, A}{\vdash_J [\Delta], \Gamma, A \oplus B} \oplus_1 \quad \frac{\vdash_J [\Delta], \Gamma, B}{\vdash_J [\Delta], \Gamma, A \oplus B} \oplus_2 \quad \text{Note } d(B) = \emptyset \text{ (resp. } d(A) = \emptyset) \text{ in } \oplus_1 \text{ (resp. } \oplus_2).$$

$\text{MALL}(I)$  has no propositional variables; every formula consists of constants. Hence, the usual identity axiom is readily derived:

**Lemma 2.1 (Identity)**  $\vdash_J A, A^\perp$  is provable for any  $\text{MALL}(I)$  formula  $A$  of domain  $J$ .

**Lemma 2.2 (Restricting provable sequents)** If  $\vdash_J [\Delta], \Gamma$  is provable, then so is  $\vdash_{J \cap K} [\Delta \upharpoonright_K], \Gamma \upharpoonright_K$  for any  $K \subseteq I$ .

## 2.2 $\text{MALL}^{[c]}(I)$ and Relational Semantics $\text{Rel}^{[c]}$

It is well known that the category  $\text{Rel}$  of relations and sets constitutes a denotational semantics of  $\text{MALL}$ , that is, the interpretation is invariant,  $(\pi_{[\Delta], \Gamma})^* = (\pi'_{[\Delta'], \Gamma})^*$ , for any reduction  $\pi_{[\Delta], \Gamma} \triangleright \pi'_{[\Delta'], \Gamma}$  of cut elimination. In particular, the denotation of  $\pi$  is equal to that of a cut-free  $\pi'$  when  $\Delta'$  is empty. The cut rule is interpreted by *relational composition* in  $\text{Rel}$ , and this interpretation makes the semantics denotational.

**Definition 2.1 (Denotational interpretation  $(\pi_{[\Delta], \Gamma})^*$  in  $\text{Rel}$ )** Every  $\text{MALL}$  proof  $\pi_{[\Delta], \Gamma}$  of a sequent  $\vdash [\Delta] \Gamma$  is interpreted by a subset of an associated set of the conclusion (without the cut list),

$$(\pi_{[\Delta], \Gamma})^* \subseteq |\Gamma|.$$

Note that in the interpretation, the cut formulas  $\Delta$  become invisible by virtue of the relational composition. We refer the reader to the definition in [Bucciarelli and Ehrhard '00]. Our aim in this paper is to investigate a dynamics of cuts hidden in such a static categorical composition. We begin by interpreting proofs in  $\text{Rel}$  but without performing cuts by means of relational composition. To stress this interpretation, the category is denoted as  $\text{Rel}^{[c]}$ , in which the cut list  $[\Delta]$  is interpreted explicitly.

To deal with the additives in  $\text{Rel}^{[c]}$ , we have to work with a sublist and the disjoint union of all the sublists:  $\hat{\Delta}$  denotes a sublist obtained by pairwise deletion (indicated by the hat) of the cut

formulas. This notation includes the empty (resp. total) deletion, hence the whole  $\Delta$  (resp. the empty list, denoted  $\Delta_\emptyset$ ) is a sublist. We define

$$\mathfrak{sl}(\Delta) := \bigsqcup_{\widehat{\Delta}} \widehat{\Delta},$$

which denotes the *disjoint union* of all the  $\widehat{\Delta}$  sublists (including  $\Delta$  and  $\Delta_\emptyset$ ).

As a result, we obtain the definition

$$|\mathfrak{sl}(\Delta)| = \sum_{\widehat{\Delta}} |\widehat{\Delta}|,$$

in which  $|\widehat{\Delta}|$ , for a nonempty sequence, is the usual interpretation of the sequence in Rel and  $|\Delta_\emptyset|$  is defined to be  $\{*\}$ , which is equal to  $|\perp|$ . The disjoint sum (in accordance with the disjoint union) is taken over different  $\widehat{\Delta}$ 's.

**Lemma 2.3** *If  $\Delta = \Delta_1, \Delta_2$  and  $\Delta_i$  are lists of pairwise-dual formulas, then*

$$|\mathfrak{sl}(\Delta)| = |\mathfrak{sl}(\Delta_1)| \times |\mathfrak{sl}(\Delta_2)|.$$

**Definition 2.2 (Interpretation  $|\pi_{[\Delta], \Gamma}|$  of proofs with unexecuted cuts in Rel<sup>[c]</sup>)** Every MALL proof  $\pi_{[\Delta], \Gamma}$  of a sequent  $\vdash [\Delta] \Gamma$  is interpreted by

$$|\pi_{[\Delta], \Gamma}| \subseteq |\mathfrak{sl}(\Delta)| \times |\Gamma|,$$

which is defined inductively and in the same manner as in Definition 2.1, except for the following: (cut rule)

$$\text{When } \pi \text{ is } \frac{\vdash [\Delta_1], \Gamma_1, A \quad \vdash [\Delta_2], A^\perp, \Gamma_2}{\vdash [\Delta_1, \Delta_2, A, A^\perp], \Gamma_1, \Gamma_2} \text{ cut}$$

$$\begin{aligned} |\pi_{[\Delta_1, \Delta_2, A, A^\perp], \Gamma_1, \Gamma_2}| &:= |\pi_{[\Delta_1], \Gamma_1, A}^1| \times |\pi_{[\Delta_2], A^\perp, \Gamma_2}^2| \subseteq |\mathfrak{sl}(\Delta_1)| \times |\Gamma_1| \times |A| \times |\mathfrak{sl}(\Delta_2)| \times |A^\perp| \times |\Gamma_2| \\ &\cong |\mathfrak{sl}(\Delta_1, \Delta_2)| \times |A| \times |A^\perp| \times |\Gamma_1| \times |\Gamma_2| \end{aligned}$$

In obtaining the last equation, Lemma 2.3 is used because the two lists  $\Delta_1$  and  $\Delta_2$  are disjoint. (&-rule)

$$\text{When } \pi \text{ is } \frac{\vdash [\Delta_1, \Sigma], \Gamma, A_1 \quad \vdash [\Delta_2, \Sigma], \Gamma, A_2}{\vdash [\Delta_1, \Delta_2, \Sigma], \Gamma, A_1 \& A_2} \&$$

$$\begin{aligned} |\pi_{[\Delta_1, \Delta_2, \Sigma], \Gamma, A_1 \& A_2}| &:= \\ \{(\lambda_1, (1, a_1)) \mid (\lambda_1, a_1) \in |\pi_{[\Delta_1, \Delta], \Gamma, A_1}^1|\} &+ \{(\lambda_2, (2, a_2)) \mid (\lambda_2, a_2) \in |\pi_{[\Delta_2, \Delta], \Gamma, A_2}^2|\} \\ \subseteq (|\mathfrak{sl}(\Delta_1, \Delta)| \times |\Gamma| \times |A_1|) &+ (|\mathfrak{sl}(\Delta_2, \Delta)| \times |\Gamma| \times |A_2|) \subseteq_\simeq |\mathfrak{sl}(\Delta_1, \Delta_2, \Delta)| \times |\Gamma| \times |A_1 \& A_2|. \end{aligned}$$

In obtaining the last equation, *monotonicity*,  $|\mathfrak{sl}(\Delta_i, \Delta)| \subseteq |\mathfrak{sl}(\Delta_1, \Delta_2, \Delta)|$ , and the *distribution* of  $\times$  over  $+$  in Rel are used.

We extend the Bucciarelli–Ehrhard translation of Definition 2.3 to Definition 2.4 to accommodate cut formulas inside the stack.

**Definition 2.3 (Translation to MALL( $I$ ) sequent  $\Gamma \langle \gamma \rangle$  [Bucciarelli and Ehrhard '00])** To any MALL formula  $A$  and a family  $a \in |A|^J$ , a formula  $A \langle a \rangle$  of MALL( $I$ ) is associated, with domain  $J$ . If  $\Gamma = A_1, \dots, A_n$  is a sequence of MALL formulas, its relational interpretation is  $|\Gamma| = |A| \times \dots \times |A_n|$ . Every  $\gamma \in |\Gamma|^J$  is written uniquely as  $\gamma = \gamma^1 \times \dots \times \gamma^n$  with  $\gamma^m \in |A_m|^J$ , and we set  $\Gamma \langle \gamma \rangle = A_1 \langle \gamma_1 \rangle \times \dots \times A_n \langle \gamma_n \rangle$ .

-For  $A = \mathbf{0}$  or  $A = T$ , if  $J \neq \emptyset$ , we have  $|A|^J = \emptyset$ , and  $A \langle a \rangle$  is undefined. If  $J = \emptyset$ ,  $|A|^J$  has exactly one element, namely, the empty family  $\emptyset$ , and we set  $\mathbf{0} \langle \emptyset \rangle = \mathbf{0}$  and  $T \langle \emptyset \rangle = T$ .

-If  $A = \mathbf{1}$  or  $A = \perp$ ,  $a$  is the constant family, and we set  $\mathbf{1} \langle (*)_J \rangle = \mathbf{1}_J$  and  $\perp \langle (*)_J \rangle = \perp_J$ .

-If  $A = B \otimes C$ , then  $a = b \times c$  with  $b \in |B|^J$  and  $c \in |C|^J$ , and we set  $A\langle a \rangle = B\langle b \rangle \otimes C\langle c \rangle$  which is a well-formed formula of  $\text{MALL}(I)$  of domain  $J$ .

Similarly, for  $A = B \wp C$  we set  $A\langle a \rangle = B\langle b \rangle \wp C\langle c \rangle$ .

-If  $A = B \oplus C$ , then  $a = b + c$  with  $b \in |B|^K$  and  $c \in |C|^L$  and  $K + L = J$ . Then we set  $A\langle a \rangle = B\langle b \rangle \oplus C\langle c \rangle$  which is a well-formed formula of  $\text{MALL}(I)$  of domain  $J$ .

Similarly for  $A = B \& C$ , we set  $A\langle a \rangle = B\langle b \rangle \& C\langle c \rangle$ .

**(Notation)**

Let  $X$  be a set and  $J = J_1 + J_2$ . Every  $x \in X^J$  yields the restrictions  $x_i = x \upharpoonright_{J_i} \in X^{J_i}$  with  $i = 1, 2$ . Conversely, the two restrictions retrieve  $x$ . We write this as  $x = x_1 \frown x_2$ .

**Definition 2.4 (Translation to  $\text{MALL}^{[c]}(I)$  sequent  $\vdash_J [\Delta\langle\delta\rangle], \Gamma\langle\gamma\rangle$ )** Let  $\Delta$  be a sequence of pairwise-dual  $\text{MALL}$  formulas and  $\delta \in |\mathfrak{sl}(\Delta)|^J$  for some  $J \subseteq I$ . Then the  $\text{MALL}(I)$  sequence  $\Delta\langle\delta\rangle$  of pairwise-dual formulas is associated such that  $d(\Delta\langle\gamma\rangle) \subseteq J$  and  $\Delta\langle\gamma\rangle \upharpoonright_\emptyset$  is  $\Delta$  as follows.

First, we write  $\Delta = \Delta_1, \dots, \Delta_n$ , where each  $\Delta_i$  is a list of two dual formulas  $A_i$  and  $A_i^\perp$ . By Lemma 2.3,  $\delta = \delta^1 \times \dots \times \delta^n$ , so  $\delta^i \in |\mathfrak{sl}(\Delta_i)|^J$ . Because all sublists of  $\Delta_i$  are  $\Delta_i$  and  $\Delta_\emptyset$ , we have  $|\mathfrak{sl}(\Delta_i)| = |A_i, A_i^\perp| + \{*\}$ . Thus,  $J$  can be divided into  $J = J_i + K_i$  to yield  $\delta^i = \delta^i \upharpoonright_{J_i} \frown \delta^i \upharpoonright_{K_i}$  so that  $\delta^i \upharpoonright_{J_i} \in |A_i, A_i^\perp|^{J_i}$  and  $\delta^i \upharpoonright_{K_i} \in |*|^{K_i}$ . Then, using the  $\delta^i \upharpoonright_{J_i}$ , we define

$$\Delta\langle\delta\rangle = \Delta_1\langle\delta^1 \upharpoonright_{J_1}\rangle, \dots, \Delta_n\langle\delta^n \upharpoonright_{J_n}\rangle,$$

in which the two dual formulas in each  $\Delta_i\langle\delta^n \upharpoonright_{J_i}\rangle$  have the same domain  $J_i \subseteq J$ .

Then, by employing Definition 2.3, for a given  $\text{MALL}$  sequent  $\vdash [\Delta], \Gamma$ , every  $\nu \in (|\mathfrak{sl}(\Delta)| \times |\Gamma|)^J$  is associated with a  $\text{MALL}(I)$  sequent, for which we write  $\nu = \delta \times \gamma$ , so that  $\delta \in |\mathfrak{sl}(\Delta)|^J$  and  $\gamma \in |\Gamma|^J$ :

$$\vdash_J ([\Delta], \Gamma)\langle\nu\rangle = \vdash_J [\Delta\langle\delta\rangle], \Gamma\langle\gamma\rangle.$$

Here  $\vdash_J ([\Delta], \Gamma)\langle\nu\rangle$  restricted to  $\emptyset$  is  $\vdash [\Delta], \Gamma$ . Note that all the formulas in  $\Gamma\langle\gamma\rangle$  have domain  $J$ , while each formula in  $\Delta\langle\delta\rangle$  has a domain contained in  $J$ . The  $\Delta\langle\delta\rangle$ 's inside the stack become a list of pairwise-dual  $\text{MALL}(I)$  formulas in which each pair has the same domain.

The translations commute with restriction of indexes, and Lemma 2.2 can be restated:

**Lemma 2.4 (Restricting translation)** - For any  $\gamma \in |\Gamma|^J$ , it holds that  $\Gamma\langle\gamma \upharpoonright_{J \cap K}\rangle = \Gamma\langle\gamma\rangle \upharpoonright_K$ ;

- For any  $\delta \in |\mathfrak{sl}(\Delta)|^J$ , it holds that  $\Delta\langle\gamma \upharpoonright_{J \cap K}\rangle = \Delta\langle\gamma\rangle \upharpoonright_K$ ;

- If  $\vdash_J [\Delta\langle\delta\rangle], \Gamma\langle\gamma\rangle$  is provable, then so is  $\vdash_{J \cap K} [\Delta\langle\delta\rangle \upharpoonright_{J \cap K}], \Gamma\langle\gamma\rangle \upharpoonright_{J \cap K}$ .

### 2.3 Fundamental lemma

The key to how the indexed linear logic arises is its tight connection to the relational semantics for the usual linear logic. The connection is realized by a fundamental lemma due to Bucciarelli & Ehrhard (proposition 20 of [Bucciarelli and Ehrhard '00]) establishing a correspondence between an indexed set in  $\text{Rel}$  and an indexed sequent in  $\text{MALL}(I)$ . The former is semantic in  $\text{MALL}$ , while the latter is syntactic in  $\text{MALL}(I)$ . This lemma is next shown to be preserved under our extended syntax and semantics, designed to accommodate cut formulas in  $\text{MALL}^{[c]}(I)$  and to  $\text{Rel}^{[c]}$ , respectively.

**Proposition 2.1 (Fundamental lemma à la Bucciarelli–Ehrhard)**

For  $\nu \in (|\mathfrak{sl}(\Delta)| \times |\Gamma|)^J$  with  $J \subseteq I$ , the following two statements are equivalent and retain a relationship  $\rho \upharpoonright_\emptyset = \pi$  between the  $\pi$  of (i) and  $\rho$  of (ii):

(i) There exists a  $\text{MALL}^{[c]}$  proof  $\pi$  such that

$$\nu \in |\pi_{[\Delta], \Gamma}|^J.$$



(ii) There exists a  $\text{MALL}^{[c]}(I)$  proof  $\rho$  to the sequent

$$\vdash_J ([\Delta], \Gamma) \langle \nu \rangle.$$

**Proof.** See lemmas B.1 and B.2 in the Appendix B.1.  $\square$

### 3 Lifting MALL reduction over indexes

This section describes how our indexed syntax  $\text{MALL}^{[c]}(I)$  analyzes Gentzen-style reduction of cut elimination for nonindexed MALL. Every MALL reduction with cut elimination is shown to be lifted to a directed transformation between two  $\text{MALL}(I)$  proofs. These transformations diminish the indexes of proofs overall.

**Definition 3.1** ( $\text{MALL}^{[c]}(I)$  proof transformation  $\blacktriangleright^{i^b}$  with diminishing indexes) A  $\text{MALL}^{[c]}(I)$  transformation  $\blacktriangleright^{i^b}$  with diminishing indexes, written as  $\rho \vdash_J [\Delta], \Gamma \blacktriangleright^{i^b} \rho' \vdash_{J'} [\Delta'], \Gamma$ , is a transformation from one  $\text{MALL}^{[c]}(I)$  proof  $\rho$  for  $\vdash_J [\Delta], \Gamma$  to another,  $\rho'$  for  $\vdash_{J'} [\Delta'], \Gamma$  with  $J' \subseteq J$ , satisfying the following two properties:

1. (Restriction to the empty domain)

Restricting the two  $\text{MALL}^{[c]}(I)$  proofs to  $\emptyset$  gives rise to a  $\text{MALL}^{[c]}$  reduction  $\pi_{[\Delta], \Gamma} \triangleright \pi'_{[\Delta'], \Gamma}$ .

Schematically, this can be written as:

$$\begin{array}{ccc} \rho \vdash_J [\Delta], \Gamma & \blacktriangleright^{i^b} & \rho' \vdash_{J'} [\Delta'], \Gamma \\ \downarrow \vdash_{\emptyset} & & \downarrow \vdash_{\emptyset} \\ \pi_{[\Delta], \Gamma} & \triangleright & \pi'_{[\Delta'], \Gamma} \end{array}$$

2. (Domain for  $\rho' \vdash_{J'} [\Delta'], \Gamma$ )

The indexes of every formula in every sequent in  $\rho' \vdash_{J'} [\Delta'], \Gamma$  are either *identical to* or *contained in* that of the corresponding formula in the corresponding sequent in  $\rho \vdash_J [\Delta], \Gamma$ . Note that each sequent (and hence formula) of  $\rho'$  has a corresponding occurrence in  $\rho$  by virtue of condition 1.

The transformation  $\rho \blacktriangleright^{i^b} \rho'$  is called a *lifting* of  $\pi \triangleright \pi'$ .

The lifting in Definition 3.1 is not unique for a given  $\text{MALL}^{[c]}$  reduction, as any subset  $J''$  of  $J'$ ,  $\rho \blacktriangleright^{i^b} \rho' \upharpoonright_{J''}$  obviously becomes a lifting for  $\rho$  and  $\rho'$  under this definition.

**Example 3.1** The following is a  $\text{MALL}^{[c]}(I)$  reduction with diminishing indexes such that its restriction to  $\emptyset$  is a Gentzen reduction to eliminate the pairwise-dual additive connectives  $\&$  and  $\oplus$  in the cut formulas:

$$\begin{array}{c} \left\{ \frac{\vdots \pi^i}{\vdash_{J_i} [\Delta_i, \Omega \upharpoonright_{J_i}], \Gamma \upharpoonright_{J_i}, A_i} \right\}_{i=1,2} \quad \& \quad \frac{\vdots \pi^3}{\vdash_{J_1+J_2} [\Delta_3], \Xi, A_1^\perp} \oplus_1 \\ \hline \vdash_{J_1+J_2} [\Delta_1, \Delta_2, \Omega], \Gamma, A_1 \& A_2 \quad \& \quad \vdash_{J_1+J_2} [\Delta_3], \Xi, A_1^\perp \oplus A_2^\perp \\ \hline \vdash_{J_1+J_2} [\Delta_1, \Delta_2, \Omega, \Delta_3, (A_1 \& A_2), (A_2^\perp \oplus A_1^\perp)], \Gamma, \Xi \quad \text{cut} \\ \\ \blacktriangleright^{i^b} \quad \frac{\vdots \pi^1}{\vdash_{J_1} [\Delta_1, \Omega], \Gamma, A_1} \quad \frac{\vdots \pi^3}{\vdash_{J_1} [\Delta_3 \upharpoonright_{J_1}], \Xi \upharpoonright_{J_1}, A_1^\perp \upharpoonright_{J_1}} \text{cut} \\ \hline \vdash_{J_1} [\Delta_1, \Omega, \Delta_3 \upharpoonright_{J_1}, A_1, A_1^\perp \upharpoonright_{J_1}], \Gamma, \Xi \end{array}$$

The indexes are diminished from  $J_1 + J_2$  to  $J_1$  as a result of erasing the subproof  $\pi^2$  with the proof transformation.

**Proposition 3.1 (Lifting to indexed transformation)** *Let  $\nu \in |\pi_{[\Delta], \Gamma}|^J$  and consider a MALL reduction  $\pi_{[\Delta], \Gamma} \triangleright \pi'_{[\Delta'], \Gamma}$ . Then there exist  $J' \subseteq J$  and  $\nu' \in |\pi'_{[\Delta'], \Gamma}|^{J'}$  to lift the given reduction:*

$$\begin{array}{ccc} \rho \vdash_J ([\Delta], \Gamma) \langle \nu \rangle & \blacktriangleright^b & \rho' \vdash_{J'} ([\Delta'], \Gamma) \langle \nu' \rangle \\ \downarrow \upharpoonright_\emptyset & & \downarrow \upharpoonright_\emptyset \\ \pi_{[\Delta], \Gamma} & \triangleright & \pi'_{[\Delta'], \Gamma} \end{array}$$

The  $\rho$  and  $\rho'$  are MALL proofs ensured by the fundamental lemma for the sequents  $\vdash_J ([\Delta], \Gamma) \langle \gamma \rangle$  and  $\vdash_{J'} ([\Delta'], \Gamma) \langle \gamma' \rangle$ , respectively. Hence, we can also denote the lifting by

$$\nu \in |\pi_{[\Delta], \Gamma}|^J \blacktriangleright^b \nu' \in |\pi'_{[\Delta'], \Gamma}|^{J'}.$$

**Proof.** For every kind of reduction  $\triangleright$ , we can directly construct  $\nu'$  together with  $J'$ . There are three crucial cases:

(Crucial case 1)

$$\frac{\overline{\vdash B, B^\perp} \quad \vdash [\Delta], A, \Gamma \overset{\vdots}{\pi'}}{\vdash [\Delta, B^\perp, A], B, \Gamma} \text{ cut}$$

(with  $A$  and  $B$  being different occurrences of the same formula) reduces to

$$\vdash [\Delta], B, \Gamma \overset{\vdots}{\pi'}$$

(identifying the occurrence of  $A$  with  $B$ ).

Let  $\epsilon \in |\text{ax}_{B, B^\perp}|^J$  and  $\tau \in |\pi'_{[\Delta], A, \Gamma}|^J$ . Then, for each  $j \in J$ , we have  $\epsilon_j = (b_j, b_j)$  with  $b_j \in |B| = |B^\perp|$ , and  $\tau_j = (\delta_j, a_j, \lambda_j)$  with  $\delta_j \in |\text{sl}(\Delta)|$ ,  $a_j \in |A|$  and  $\lambda_j \in |\Gamma|$ . Note that  $\nu_j = (\delta_j, b_j, a_j, b_j, \lambda_j)$ . We define

$$J' = \{j \in J \mid b_j = a_j\} \quad \text{and} \quad \nu' = \tau \upharpoonright_{J'}.$$

(Crucial case 2)

This case is a reduction of Example 3.1 above, with restriction to the empty domain  $\emptyset$  and identification of  $\text{MALL}(\emptyset)$  with MALL.

By  $\pi$ 's last rule,  $\nu \simeq \tau \times \lambda$ , so  $\tau \in |\&(\pi^1, \pi^2)|^J$  and  $\lambda \in |\oplus_1(\pi^3)|^J$ . Then the  $\&$ -rule of the left premise divides  $J$  into  $J = J_1 + J_2$ . We define

$$J' = J_1 \quad \text{and} \quad \nu' = \tau \upharpoonright_{J_1} \times \lambda', \quad \text{where } \lambda' = \{(x, a)_j \mid (x, (1, a))_j = \lambda_j\}.$$

(Crucial case 3) Here

$$\frac{\vdash [\Xi], \Delta, A \overset{\vdots}{\rho} \quad \left\{ \vdash [\Sigma_i, \Omega], A^\perp, \Gamma, B_i \overset{\vdots}{\pi^i} \right\}_{i=1,2}}{\vdash [\Xi, \Sigma_1, \Sigma_2, \Omega, A, A^\perp], \Delta, \Gamma, B_1 \& B_2} \& \text{ cut}$$

reduces to

$$\frac{\left\{ \frac{\vdash [\Xi], \Delta, A \overset{\vdots}{\rho} \quad \vdash [\Sigma_i, \Omega], A^\perp, \Gamma, B_i \overset{\vdots}{\pi^i}}{\vdash [\Xi, \Sigma_i, \Omega, A, A^\perp], \Delta, \Gamma, B_i} \text{ cut} \right\}_{i=1,2}}{\vdash [\Sigma_1, \Sigma_2, A, A^\perp, A, A^\perp, \Xi, \Omega], \Delta, \Gamma, B_1 \& B_2} \&$$

Note that in the last  $\&$ -rule of  $\pi'$ ,  $\Xi$  and  $\Omega$  inside the stack are chosen to be superposed.

By  $\pi$ 's last rule,  $\nu \simeq \lambda \times \tau$ , so  $\lambda \in |\rho_{[\Xi], \Delta, A]}^J$  and  $\tau \in |\&(\pi^1, \pi^2)|^J$ . The last  $\&$ -rule of the

right premise divides  $J$  into  $J = J_1 + J_2$  so that  $\tau \simeq \tau_1 \frown \tau_2$  and  $\tau_i \in |\pi_{[\Sigma_i, \Omega], A^\perp, \Gamma, B_i}^i|^{J_i}$ . Then  $\lambda \upharpoonright_{J_i} \times \tau_i \in \simeq | \text{cut}(\rho, \pi^i) |^{J_i}$ . We define  $J' = J$  and  $\nu' \simeq (\lambda \upharpoonright_{J_1} \times \tau_1) \frown (\lambda \upharpoonright_{J_2} \times \tau_2)$ .

□

## 4 MALL GoI Interpretation

### 4.1 Execution formula with zero action on symmetries of cuts

Our category framework is a minimal part of the Haghverdi–Scott GoI situation [Haghverdi Scott '06] with a reflexive object  $U$  in a traced symmetric monoidal category  $\mathcal{C}$ . Our framework in addition requires that  $\mathcal{C}$  has zero morphisms, in particular, a zero endomorphism  $0_U$  on  $U$ :

$$(\mathcal{C}, \otimes, s_{U,U}, j : U \otimes U \triangleleft U : k, 0_U)$$

$s_{U,U}$  is a symmetry  $U \otimes U \longrightarrow U \otimes U$  of tensor product.  $j : U^2 \triangleleft U : k$  denotes a pair of morphisms  $j$  and  $k$  respectively from  $U^2$  to  $U$  and the other way around.  $j$  and  $k$  are called respectively *co-retraction* and *retraction* for the reflexive  $U$  so that  $k \circ j = U \otimes U = U^2$ . The  $m$ -ary tensor folding  $\underbrace{X \otimes \cdots \otimes X}_m$  is denoted by  $X^m$  both for object  $X$  or morphism  $X$ . The trace structure will be introduced later in (11).

The zero endomorphism  $0_U$  acts conjugately (both precomposing and composing) on the symmetry  $s$  as follows:

**Definition 4.1 (zero-action  $(s_{U,U})^0$ )** 0 action on the symmetry  $s_{U,U}$  on  $U^2$  is defined to *annihilate* the symmetry  $s_{U,U}$  to the zero endomorphism on  $U^2$ .

$$(s_{U,U})^0 := 0_{U \otimes U} \circ s_{U,U} \circ 0_{U \otimes U} = (0_U \otimes 0_U) \circ s_{U,U} \circ (0_U \otimes 0_U) = 0_U \otimes 0_U = 0_{U \otimes U}$$

The second last equation is the absorbing property of the zero (w.r.t composition), and the last equation is the uniqueness of zero morphism.

We abbreviate  $s_{U,U}$  and  $(s_{U,U})^0$  as  $s$  and  $s^0$ . To avoid collapsing the categorical framework, we assume

$$s \text{ is nonzero; that is, } s \text{ and } s^0 \text{ are distinguishable endomorphisms on } U^2 \text{ in } \mathcal{C}. \quad (5)$$

The zero morphism, which is required in our framework, exists in crucial examples of GoI situations: (i)  $\text{Rel}_+$  is the  $\text{Rel}$  with  $\otimes = +$  and a reflexive object  $\mathbb{N}$ . The *empty relation* on  $\mathbb{N}$  is the zero morphism. (ii) The monoidal subcategories  $\text{Pfn}$  and  $\text{Plnj}$  of  $\text{Rel}_+$ , both retain the zero morphism.  $\text{Plnj}$  is known to be equivalent to the original category  $\text{Hilb}_2$  of Hilbert spaces and partial isometries for Girard's GoI I [Girard '89].

( $\mathbf{x} \in |\pi_{[\Delta], \Gamma}|$  as a forest) Every constituent  $x_i$  of the tuple  $\mathbf{x} = (x_1, \dots, x_\ell) \in |\pi_{[\Delta], \Gamma}| \subseteq |\mathfrak{sl}(\Delta)| \times |\Gamma|$  in Definition 2.2 belongs to the class defined by the BNF grammar defining the tree structure as follows:

$$x ::= * \mid (x, x) \mid (1, x) \mid (2, x) \quad (6)$$

Every constituent  $x_i$  is considered as a *rooted tree*, hence  $\mathbf{x} = (x_1, \dots, x_\ell)$  as a *forest* of the union of the constituent trees  $x_i$ 's.

In what follows, we make the permutation  $(x_{\tau(1)}, \dots, x_{\tau(\ell)})$  among the constituents implicit so that  $\mathbf{x}$  is up to the permutation, denoted by “ $\simeq$ ”. This is because the permutation corresponds to the exchange rule eliminated from our syntax.

**Definition 4.2 (Endomorphism  $\llbracket \mathbf{x} \rrbracket$  on tensor-folding  $U$ 's and tensor-folding  $\sigma_{\mathbf{x}}$  of symmetry  $s$  and zero  $s^0$ )** Every  $\mathbf{x} = (x_1, \dots, x_\ell) \in |\pi_{[\Delta], \Gamma}|$  is interpreted as an endomorphism  $\llbracket \mathbf{x} \rrbracket$  on the tensor product  $U^\ell$  together with an endomorphism  $\sigma_{\mathbf{x}}$  on a subproduct  $U^{2m}$  of  $U^\ell$ . The endomorphism  $\sigma_{\mathbf{x}}$  interprets cut rules in  $\pi_{[\Delta], \Gamma}$  and is  $m$ -ary tensor folding each of which component is either  $s^1 = s$  or  $s^0$  (cf. Definition 4.1) both on  $U^2$ :

$$\llbracket \mathbf{x} \rrbracket : U^\ell \longrightarrow U^\ell \quad \text{and} \quad \sigma_{\mathbf{x}} = \otimes_{i=1}^m s^{\eta(i)} \quad \text{where } \eta \text{ is a } \{0, 1\}\text{-valued function.} \quad (7)$$

The  $i$ -th component  $U$  of  $U^\ell$  is considered indexed by the  $i$ -th constituent  $x_i$ , written  $U_{x_i}$ . Since each  $x_i$  has a unique corresponding occurrence of formula  $A$  in  $\vdash [\Delta], \Gamma$  so that  $x_i \in |A|$ . The component  $U_{x_i}$  is also written  $U_A$  indexed by  $A$ . The subproduct  $U^{2m}$  arises as tensor folding of  $U_A \otimes U_{A^\perp}$ 's for certain pairs of cut formulas  $A, A^\perp$  from a sublist  $\widehat{\Delta}$  of  $\Delta$ . Then, each component  $s^{\ell(i)}$  of  $\sigma_{\mathbf{x}}$  is specified, by the Kronecker delta  $\delta$ , to be an endomorphism  $s^{\delta_{a,a'}}$  on  $U_a \otimes U_{a'}$  for paired two constituents  $a \in |A|$  and  $a' \in |A^\perp|$  of  $\mathbf{x}$ . That is

$$\sigma_{\mathbf{x}} = \otimes s^{\delta_{a,a'}} \quad \text{where } \otimes \text{ ranges over paired two constituents } a \text{ and } a' \text{ in } \mathbf{x}. \quad (8)$$

We define  $(\llbracket \mathbf{x} \rrbracket, \sigma_{\mathbf{x}})$  by induction on the construction of the proof  $\pi$ .

In the definition, each of  $\mathbf{y}, \mathbf{y}^i, \mathbf{z}, \mathbf{z}^i$  denotes a sequence of constituent defined in the BNF (6).

(Axiom)

$\mathbf{x} = (\ast, \ast) \in |\vdash A^\perp, A|$  with  $A = 1$  and  $A^\perp = \perp$ .

We define  $\llbracket \mathbf{x} \rrbracket$  to be a symmetry  $s_{U_{\ast}, U_{\ast}}$  on  $U_{\ast} \otimes U_{\ast}$  of  $\mathcal{C}$ . Because  $\pi$  is cut-free,  $\sigma_{\mathbf{x}}$  is empty by definition.

(Cut rule)

$\mathbf{x} = (\mathbf{y}^1, z^1, a, a', \mathbf{y}^2, z^2) \in |\widehat{\Delta}_1| \times |\widehat{\Delta}_2| \times |A| \times |A^\perp| \times |\Gamma_1| \times |\Gamma_2|$ , so  $\mathbf{y} = (\mathbf{y}^1, \mathbf{y}^2, a)$  and  $\mathbf{z} = (z^1, a', z^2)$  arise respectively from  $|left\ premise|$  and  $|right\ premise|$ .

We also define

$$\llbracket \mathbf{x} \rrbracket \cong \llbracket \mathbf{y} \rrbracket \otimes \llbracket \mathbf{z} \rrbracket \quad \text{and} \quad \sigma_{\mathbf{x}} \text{ is } (s_{U_a, U_{a'}})^{\delta_{a,a'}} \text{ extended to } \sigma_{\mathbf{y}} \otimes \sigma_{\mathbf{z}}$$

That is, let  $s$  denote  $s_{U_a, U_{a'}}$ , if  $a = a'$  (resp. else), then  $\sigma_{\mathbf{x}}$  is  $s$  (resp.  $s^0$ ) extended to  $\sigma_{\mathbf{y}} \otimes \sigma_{\mathbf{z}}$ . Note the extension makes sense because  $\sigma_{\mathbf{y}} \otimes \sigma_{\mathbf{z}}$  acts on the disjoint domain both with  $U_a$  and  $U_{a'}$ .

We say the cut matches (resp. mismatches) in  $\mathbf{x}$  if  $a = a'$  (resp. otherwise).

( $\wp$ -rule)

$\mathbf{x} = (\mathbf{y}, (a, b))$ , so that  $\mathbf{x}' = (\mathbf{y}, a, b) \in |premise|$ .  $\llbracket \mathbf{x} \rrbracket$  is obtained directly from  $\llbracket \mathbf{x}' \rrbracket$  on  $U^{\ell+1}$  by the retraction  $U_a \otimes U_b \triangleleft U_{(a,b)}$ . That is,  $\llbracket \mathbf{x} \rrbracket = \llbracket \mathbf{x}' \rrbracket^{(j,k)} = (U^{\ell-1} \otimes j) \circ \llbracket \mathbf{x}' \rrbracket \circ (U^{\ell-1} \otimes k)$ . We also define  $\sigma_{\mathbf{x}}$  is  $\sigma_{\mathbf{x}'}$ .

( $\otimes$ -rule)

$\mathbf{x} = (\mathbf{y}^1, z^1, \mathbf{y}^2, z^2, (a, b)) \in |\widehat{\Delta}_1| \times |\widehat{\Delta}_2| \times |\Gamma_1| \times |\Gamma_2| \times |A| \times |B|$ , so that  $\mathbf{y} = (\mathbf{y}^1, \mathbf{y}^2, a)$  and  $\mathbf{z} = (z^1, z^2, b)$  are respectively from  $|left\ premise|$  and  $|right\ premise|$ ;  $\llbracket \mathbf{x} \rrbracket$  is obtained directly from  $\llbracket \mathbf{y} \rrbracket \otimes \llbracket \mathbf{z} \rrbracket$  on  $U^{\ell+1}$  by the retraction  $U_a \otimes U_b \triangleleft U_{(a,b)}$ . That is,  $\llbracket \mathbf{x} \rrbracket = (\llbracket \mathbf{y} \rrbracket \otimes \llbracket \mathbf{z} \rrbracket)^{(j,k)} = (U^{\ell-1} \otimes j) \circ (\llbracket \mathbf{y} \rrbracket \otimes \llbracket \mathbf{z} \rrbracket) \circ (U^{\ell-1} \otimes k)$ . We also define  $\sigma_{\mathbf{x}}$  is  $\sigma_{\mathbf{y}} \otimes \sigma_{\mathbf{z}}$ .

( $\&$ -rule)

The quantity  $\mathbf{x}$  is either  $(\mathbf{y}, (1, a))$  or  $(\mathbf{y}, (2, a))$ , so that the  $(\mathbf{y}, a)$  are respectively from either  $|left\ premise|$  or  $|right\ premise|$ . We define  $\llbracket \mathbf{x} \rrbracket = \llbracket (\mathbf{y}, a) \rrbracket$  and  $\sigma_{\mathbf{x}} = \sigma_{(\mathbf{y}, a)}$  by identifying  $U_{(1,a)} = U_{(2,a)} = U_a$ .

( $\oplus_i$ -rule)

Same as  $\&$ -rule.

(**Note**) The retraction and the co-retraction are used only for multiplicatives, either in the  $\wp$ -rule or the  $\otimes$ -rule, in which  $\llbracket \mathbf{x} \rrbracket$  was constructed respectively  $\llbracket \mathbf{x}' \rrbracket^{(j,k)}$  or  $(\llbracket \mathbf{x}_1 \rrbracket \otimes \llbracket \mathbf{x}_2 \rrbracket)^{(j,k)}$ . The co-retraction  $j$  and the retraction  $k$  are always used pair-wisely to produced one component  $U$  respectively in the co-domain and in the domain of  $\llbracket \mathbf{x} \rrbracket$ .

(The endomorphism  $\llbracket \mathbf{x} \rrbracket$  as I/O box)

The endomorphism  $\llbracket \mathbf{x} \rrbracket$  is seen as an input/output (I/O) box on  $(n + 2m)$ -ary tensor  $U$ , whose inputs/outputs are the formulas occurring in  $\Gamma, \hat{\Delta}$ , in which  $\Gamma$  contains  $n$  occurrences of formulas, and a sublist  $\hat{\Delta}$  contains  $2m$  occurrences of formulas. More special box for  $\sigma_{\mathbf{x}}$ , consisting of  $m$ -ary tensor folding of  $\{s, s^0\}$  for the I/O formulas of the sublist  $\hat{\Delta}$ .

Definition 4.2 will be accompanied by action  $\epsilon_{\mathbf{x}}$  to be defined in Definition 4.5 below. The action  $\epsilon_{\mathbf{x}}$  annihilates, in terms of the zero morphism  $0_U$ , certain class of retractions and co-retractions used in constructing  $\llbracket \mathbf{x} \rrbracket$ . The class consists of associated retraction (resp. co-retraction), defined Definition 4.3, which is a retraction  $\triangleright$  (resp. co-retraction  $\triangleleft$ ) used in Definition 4.2 to obtain a component  $U_{x_i}$  (called, contracted component) of the domain (resp. of the co-domain) of  $\llbracket \mathbf{x} \rrbracket$  in the way  $U_{x_i} \triangleright U \otimes U$  (resp.  $U \otimes U \triangleleft U_{x_i}$ ).

**Definition 4.3 (associated retraction (resp. co-retraction) with contracted component of the domain (resp. co-domain) of  $\llbracket \mathbf{x} \rrbracket$ )** Let  $\mathbf{x} = (x_1, \dots, x_\ell) \in |\pi_{[\Delta], \Gamma}|$ , and consider a component  $U_{x_i}$  of the co-domain (resp. domain) of  $\llbracket \mathbf{x} \rrbracket$  for  $x_i \in |\text{a formula in } \Gamma|$ . A component  $U_{x_i}$  is called *contracted* inductively on the construction  $\pi$ <sup>1</sup>. Every contracted component in the domain (resp. co-domain) of  $\llbracket \mathbf{x} \rrbracket$  is a domain (resp. co-domain) of a unique retraction (resp. co-retraction), called *associated retraction* (resp. *associated co-retraction*)<sup>2</sup>, which was used in Definition 4.2 to interpret the multiplicative ( $\wp$  and  $\otimes$ ) rules.

(Axiom)  $\mathbf{x}$  has no contracted component so that neither  $U_*$  nor  $U_*$  are contracted.

(Multiplicatives  $\otimes$  and  $\wp$ ) The introduced  $U_{(a,b)}$  in the domain (resp. co-domain) is a contracted component, and the asret (resp. ascoret) is  $k : U_{(a,b)} \triangleright U_a \otimes U_b$  (resp.  $j : U_a \otimes U_b \triangleleft U_{(a,b)}$ ). Other contracted components are those of  $\mathbf{y}$  and  $\mathbf{z}$  except  $U_a$  and  $U_b$ .

(Additives  $\&$  and  $\oplus_i$ ) Contracted components are those of  $(\mathbf{y}, \mathbf{z})$  under the identification of  $U_{(i,z)}$  with  $U_z$  for  $i = 1, 2$ .

(Asret's and ascoret's in I/O box  $\llbracket \mathbf{x} \rrbracket$ )

When the endomorphism  $\llbracket \mathbf{x} \rrbracket$  is seen as the I/O box, the asret's and the ascoret's are those  $\triangleright$ 's and  $\triangleleft$ 's whose domains and co-domains lie respectively among the inputs and among the outputs of  $\llbracket \mathbf{x} \rrbracket$ . By the construction of  $\llbracket \mathbf{x} \rrbracket$ , they lie pair-wisely in the inputs and in the outputs. See Figure 1 (upper-left) for  $\llbracket \mathbf{x} \rrbracket$  depicting the occurrence of the asret's  $\triangleright$ 's and the ascoret's  $\triangleleft$ 's.

Although  $\triangleleft \circ \triangleright$  is not an identity on  $U$  in general, it composes (resp. precomposes) to any retraction (resp. co-retraction) identically. Thus, when  $\triangleleft \circ \triangleright$  is composed (resp. precomposed) with any contracted component in the co-domain (resp. domain) of  $\llbracket \mathbf{x} \rrbracket$ , the following holds;

$$\triangleleft \circ \triangleright \circ \llbracket \mathbf{x} \rrbracket = \llbracket \mathbf{x} \rrbracket \quad (\text{resp.} \quad \llbracket \mathbf{x} \rrbracket \circ \triangleleft \circ \triangleright = \llbracket \mathbf{x} \rrbracket) \quad (9)$$

(Convention) In (9) we take a convention that composing (resp. precomposing)  $\triangleleft \circ \triangleright$  is to an indicated ascoret (resp. asret) so that the Id is omitted on the other components of  $\llbracket \mathbf{x} \rrbracket$ 's co-domain (resp. domain). In what follows, the same convention is employed when morphisms have indicated occurrences of asret's and ascoret's.

In this equation (9), the ascoret (resp. asret) occurs explicitly as the last composed  $\triangleleft$  (resp. the first precomposed  $\triangleright$ ). Since every contracted component occurs pairwise in the co-domain and the domain of  $\llbracket \mathbf{x} \rrbracket$ , the two equations are pairwise, hence written successively at once;

$$\triangleleft \circ \triangleright \circ \llbracket \mathbf{x} \rrbracket \circ \triangleleft \circ \triangleright = \llbracket \mathbf{x} \rrbracket$$

The asret's and the ascoret's are separate w.r.t different components. Thus, for all the plural pairs of contracted components in the domain and the co-domain of  $\llbracket \mathbf{x} \rrbracket$ , the simultaneous composition and precomposition with  $\triangleleft \circ \triangleright$ s are realized by composing and precomposing the  $r$ -ary tensor folding  $(\triangleleft \circ \triangleright)^r = \triangleleft^r \circ \triangleright^r$  for a certain natural number  $r$ , hence  $\llbracket \mathbf{x} \rrbracket$  is written so that all the asret's  $\triangleright^r$  and the ascoret's  $\triangleleft^r$  are explicitly represented;

$$\llbracket \mathbf{x} \rrbracket = \triangleleft^r \circ \llbracket \mathbf{x} \rrbracket \circ \triangleright^r \quad \text{where} \quad \llbracket \mathbf{x} \rrbracket^\circ := \triangleright^r \circ \llbracket \mathbf{x} \rrbracket \circ \triangleleft^r \quad (10)$$

<sup>1</sup> For the choice of  $x_i$  (i.e., a choice of a formula (not in  $\Delta$  but) in  $\Gamma$ ), the construction is free from cut.

<sup>2</sup> asret (resp. ascoret) for short

$\llbracket \mathbf{x} \rrbracket$  and  $\llbracket \mathbf{x} \rrbracket^\circ$  are mutually obtained one another.

In what follows, we shall see how feedback stemming from Gentzen cut-elimination for a MALL proof  $\pi$  acts on the asret's and the ascot's of  $\llbracket \mathbf{x} \rrbracket$  for  $\mathbf{x} \in |\pi_{[\Delta], \Gamma}|$ . The action is stipulated in terms of the zero morphism added in our framework. First, in a categorical framework of Girard's GoI project, the feedback is modelled by the trace structure (cf. [Joyal *et al.* '96]) of the following natural family satisfying the certain axioms:

$$\mathrm{Tr}_{X,Y}^Z : \mathcal{C}(X \otimes Z, Y \otimes Z) \longrightarrow \mathcal{C}(X, Y) \quad (11)$$

The naturality has three kinds, *naturality* in  $X$  and *naturality* in  $Y$ , and *dinaturality* in  $Z$ . The axioms are *vanishing*, *superposing* and *yanking*. See Appendix A.1 for the three naturalities and the three axioms.

In our setting of Definition 4.2, the endomorphism  $\llbracket \mathbf{x} \rrbracket$  is on  $U^{n+2m}$  so that  $n$  and  $2m$  are the numbers of formulas respectively in  $\Gamma$  and in a sublist  $\Delta$ , and  $\sigma_{\mathbf{x}}$  is on the subproduct  $U^{2m}$ . Hence the feedback is calculated by;

$$\mathrm{ex}(\sigma, \mathbf{x}) := \mathrm{Tr}_{U^n, U^n}^{U^{2m}} ((\mathrm{Id} \otimes \sigma_{\mathbf{x}}) \circ \llbracket \mathbf{x} \rrbracket) \quad (12)$$

Note that when  $\mathbf{x} \in |\pi_{[\Delta], \Gamma}|$  comes from a proof  $\pi$  of the multiplicative fragment, the equation is exactly the GoI interpretation of the proof  $\pi_{[\Delta], \Gamma}$  (cf. [Haghverdi Scott '06]). This is because in the multiplicative fragment, the index  $I$  becomes redundantly the singleton  $\{*\}$ , thus  $|\pi_{[\Delta], \Gamma}| = \{\mathbf{x}\}$ , whereby  $\sigma_{\mathbf{x}}$  consists only of the symmetry  $s$ .

By naturalities of traces, the ascot's (resp. the asret's) of  $\llbracket \mathbf{x} \rrbracket$  retain one-to-one corresponding occurrences in  $\mathrm{Tr}_{U^n, U^n}^{U^{2m}} ((\mathrm{Id} \otimes \sigma_{\mathbf{x}}) \circ \llbracket \mathbf{x} \rrbracket)$ ; For tracing (9) yields;

$$\begin{aligned} \mathrm{Tr}_{U^n, U^n}^{U^{2m}} ((\mathrm{Id} \otimes \sigma_{\mathbf{x}}) \circ (\triangleleft \circ \triangleright \circ \llbracket \mathbf{x} \rrbracket)) &= \triangleleft \circ \triangleright \circ \mathrm{ex}(\sigma, \mathbf{x}) \\ (\text{resp. } \mathrm{Tr}_{U^n, U^n}^{U^{2m}} ((\mathrm{Id} \otimes \sigma_{\mathbf{x}}) \circ (\llbracket \mathbf{x} \rrbracket \circ \triangleleft \circ \triangleright))) &= \mathrm{ex}(\sigma, \mathbf{x}) \triangleleft \circ \triangleright. \end{aligned} \quad (13)$$

Thus all the asret's and the ascot's of  $\llbracket \mathbf{x} \rrbracket$  are written explicitly;

$$\mathrm{Tr}_{U^n, U^n}^{U^{2m}} ((\mathrm{Id} \otimes \sigma_{\mathbf{x}}) \circ \llbracket \mathbf{x} \rrbracket) = \triangleleft^r \circ \mathrm{Tr}_{U^n, U^n}^{U^{2m}} ((\mathrm{Id} \otimes \sigma_{\mathbf{x}}) \circ \llbracket \mathbf{x} \rrbracket^\circ) \circ \triangleright^r \quad (14)$$

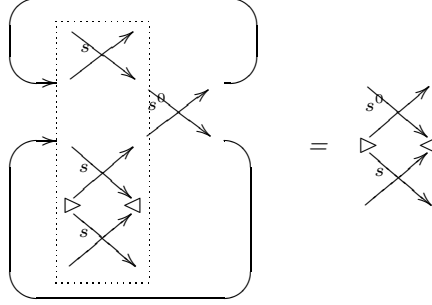
While inside the sole  $\llbracket \mathbf{x} \rrbracket$ , the asret's and the ascot's (written explicitly in (10)) do not interact with zero morphisms because the constriction  $\llbracket \mathbf{x} \rrbracket$  of Definition 4.2 is free from the zero morphisms. Remind that the zero morphisms live only in  $\sigma_{\mathbf{x}}$  (cf.(7)). However when they are put inside the context  $\mathrm{Tr}_{U^n, U^n}^{U^{2m}} ((\mathrm{Id} \otimes \sigma_{\mathbf{x}}) \circ -)$  (written explicitly in (14)), they may interact with zero morphisms arisen from  $\sigma_{\mathbf{x}}$  via the feedback of the trace. That is, the tracing in monoidal category takes feedback into account, hence yields interacting the zeros stemming from  $\sigma_{\mathbf{x}}$ . This yields a certain action on the asret's and the ascot's of  $\llbracket \mathbf{x} \rrbracket$ , as defined in Definition 4.5 below. We begin with

**Definition 4.4 (zero input (resp. output) of ascot (resp. asret) w.r.t the interpretation  $\sigma_{\mathbf{x}}$  of cuts)**

**(zero input of ascot  $\triangleleft$ )** An ascot  $\triangleleft$  of  $\llbracket \mathbf{x} \rrbracket$  is called to have *zero input* w.r.t  $\sigma_{\mathbf{x}}$  when  $\triangleleft$  decomposes in  $\mathrm{ex}(\sigma, \mathbf{x})$  either  $\triangleleft \circ (0_U \otimes U)$  or  $\triangleleft \circ (U \otimes 0_U)$ . That is, when the ascot is written explicitly as  $\mathrm{ex}(\sigma, \mathbf{x}) = \triangleleft \circ \mathbf{g}$  so that  $\mathbf{g} = \triangleright \circ \mathrm{ex}(\sigma, \mathbf{x})$  (cf. (9)), either  $0_U \otimes U$  or  $U \otimes 0_U$  acts identically on  $\mathbf{g}$  by composing to the indicated component  $U \otimes U$ .

**(zero output of asret  $\triangleright$ )** An asret  $\triangleright$  of  $\llbracket \mathbf{x} \rrbracket$  is called to have *zero output* w.r.t  $\sigma_{\mathbf{x}}$  when  $\triangleright$  decomposes in  $\mathrm{ex}(\sigma, \mathbf{x})$  either  $(0_U \otimes U) \circ \triangleright$  or  $(U \otimes 0_U) \circ \triangleright$ . That is, when the asret is written explicitly as  $\mathrm{ex}(\sigma, \mathbf{x}) = \mathbf{g} \circ \triangleright$  so that  $\mathbf{g} = \mathrm{ex}(\sigma, \llbracket \mathbf{x} \rrbracket) \circ \triangleleft$  (cf. (9)), either  $0_U \otimes U$  or  $U \otimes 0_U$  acts identically on  $\mathbf{g}$  by precomposing to the indicated component  $U \otimes U$ .

Why do we call zero input (resp. output) ? The  $U \otimes U$  of  $\triangleleft$ 's domain (resp.  $\triangleright$ 's co-domain) can be regarded having two inputs (resp. outputs), one left component  $U$  and the another right one. Then the decomposition in each case says that one of two inputs (resp. outputs) is zero.



**Example 4.1** *Let*

$\pi[A \& A, A^\perp \oplus A^\perp] \ A^\perp, A \otimes B, B^\perp$  be a proof obtained by a  $\otimes$ -rule between  $\pi_1$  of Section 1.1 and  $ax_{B^\perp, B}$ . Let  $\mathbf{x} := \nu_2 \times (\bar{\star}, \star) \in |\pi|$ , where  $\nu_2$  is in Section 1.1 and  $(\bar{\star}, \star) \in |ax_{B^\perp, B}|$ . Then  $\llbracket \mathbf{x} \rrbracket$  has the unique pair of asret and ascoret both interpreting the  $\otimes$ -rule. See the left-hand dotted rectangle repre-

senting  $\llbracket \mathbf{x} \rrbracket$  with the asret and the ascoret. The pair of asret and the ascoret appears explicitly in the second and the third of the following equations (in which  $\sigma_{\mathbf{x}} = s_{U,U}^0$ ):

$\text{Tr}_{U^3, U^3}^{U^2} ((\text{Id} \otimes \sigma_{\mathbf{x}}) \circ \llbracket \mathbf{x} \rrbracket) = \triangleleft \circ \text{Tr}_{U^4, U^4}^{U^2} ((\text{Id} \otimes \sigma_{\mathbf{x}}) \circ \llbracket \mathbf{x} \rrbracket^0) \circ \triangleright = \triangleleft \circ (s_{U,U}^0 \otimes s_{U,U}) \circ \triangleright$ .  
See the above figure whose LHS and RHS are the first and the last eqns, respectively. The ascoret (resp. asret) has zero input (resp. output) because the right picture depicts  $\triangleleft$  (resp.  $\triangleright$ ) having a zero input (resp. output) from the northwest (resp. to the northeast) so that  $(0_U \otimes U)$  composes (resp. precomposes) to  $s_{U,U}^0 \otimes s_{U,U}$  identically.

**Definition 4.5 (action  $\epsilon_{\mathbf{x}}$  on asret's and ascoret's of  $\llbracket \mathbf{x} \rrbracket$ )** The endomorphism  $\sigma_{\mathbf{x}}$  of Definition 4.2 for  $\mathbf{x} \in |\pi_{[\Delta], \Gamma}|$  yields the following action  $\epsilon_{\mathbf{x}}$  on the asret's and the ascoret's of  $\llbracket \mathbf{x} \rrbracket$ . The  $\epsilon_{\mathbf{x}}$  acts each asret and ascoret *either zero or identical*.

$$\triangleright^{\epsilon_{\mathbf{x}}} = \begin{cases} \triangleright^0 & \text{if } \triangleright \text{ has a zero-output} \\ \triangleright & \text{otherwise} \end{cases} \quad \text{w.r.t } \sigma_{\mathbf{x}} \quad \triangleleft^{\epsilon_{\mathbf{x}}} = \begin{cases} \triangleleft^0 & \text{if } \triangleleft \text{ has a zero input} \\ \triangleleft & \text{otherwise} \end{cases} \quad \text{w.r.t } \sigma_{\mathbf{x}}$$

where *zero actions*  $\triangleright^0$  and  $\triangleleft^0$  are defined respectively as follows:

$$\triangleright^0 := 0_{U, U \otimes U} = k \circ 0_U = (0_U \otimes 0_U) \circ k \quad \triangleleft^0 := 0_{U \otimes U, U} = 0_U \circ j = j \circ (0_U \otimes 0_U)$$

That is, the zero *annihilate* the pair of asret and ascoret  $j : U \otimes U \triangleleft U : k$  to the pair of the zero morphisms  $j^0 : U \otimes U \triangleleft^0 U : k^0$ , where  $j^0 = \triangleleft^0$  and  $k^0 = \triangleright^0$ .

The action  $\epsilon_{\mathbf{x}}$  of Definition 4.5 is conjugate on the pairwise tensor folding  $(\triangleright^r, \triangleleft^r)$  of the asret's and the ascoret's, so we may write it acting on  $\llbracket \mathbf{x} \rrbracket$  conjugately;

$$\llbracket \mathbf{x} \rrbracket^{\epsilon_{\mathbf{x}}} := (\triangleleft^{\epsilon_{\mathbf{x}}})^r \circ \llbracket \mathbf{x} \rrbracket \circ (\triangleright^{\epsilon_{\mathbf{x}}})^r \quad (15)$$

This, by naturalities, extends to the action  $\epsilon_{\mathbf{x}}$  on the corresponding retractions and co-retractions in (14);

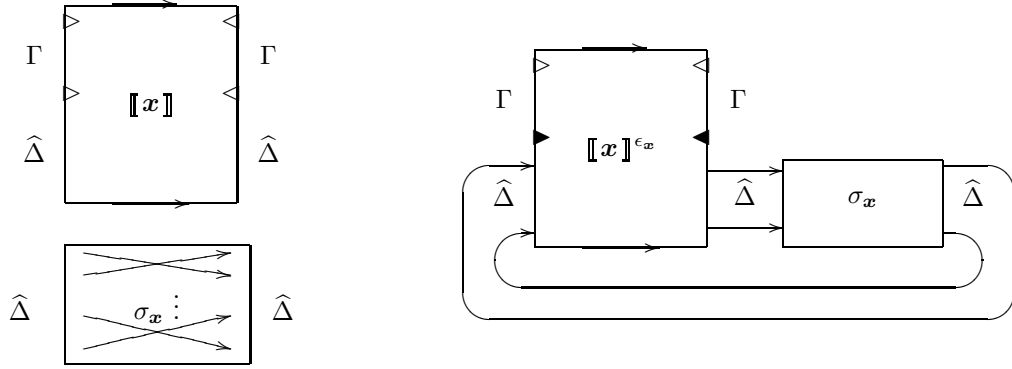
$$\begin{aligned} (\text{Tr}_{U^n, U^n}^{U^{2m}} ((\text{Id} \otimes \sigma_{\mathbf{x}}) \circ \llbracket \mathbf{x} \rrbracket))^{\epsilon_{\mathbf{x}}} &:= (\triangleleft^{\epsilon_{\mathbf{x}}})^r \circ \text{Tr}_{U^n, U^n}^{U^{2m}} ((\text{Id} \otimes \sigma_{\mathbf{x}}) \circ \llbracket \mathbf{x} \rrbracket) \circ (\triangleright^{\epsilon_{\mathbf{x}}})^r \\ &= \text{Tr}_{U^n, U^n}^{U^{2m}} ((\text{Id} \otimes \sigma_{\mathbf{x}}) \circ \llbracket \mathbf{x} \rrbracket^{\epsilon_{\mathbf{x}}}) \end{aligned}$$

**Definition 4.6 (Execution formula  $\text{Ex}(\sigma, \mathbf{x})$  for  $\mathbf{x} \in |\pi_{[\Delta], \Gamma}|$ )**

For every  $\mathbf{x} \in |\pi_{[\Delta], \Gamma}|$ , the endomorphism  $\text{Ex}(\sigma, \mathbf{x})$  is defined

$$\begin{aligned} \text{Ex}(\sigma, \mathbf{x}) &:= \text{ex}(\sigma, \mathbf{x})^{\epsilon_{\mathbf{x}}} \\ &= \text{Tr}_{U^n, U^n}^{U^{2m}} ((\text{Id} \otimes \sigma_{\mathbf{x}}) \circ \llbracket \mathbf{x} \rrbracket^{\epsilon_{\mathbf{x}}}), \end{aligned}$$

where  $(\llbracket \mathbf{x} \rrbracket, \sigma_{\mathbf{x}})$  is the pair of the endomorphism on  $U^{n+2m}$  and on the subproduct  $U^{2m}$  in Definition 4.2 and  $\epsilon_{\mathbf{x}}$  is the action in Definition 4.5 on the asret's and the ascoret's of  $\llbracket \mathbf{x} \rrbracket$ . The domains (resp. the co-domains) of the asret's (resp. the ascoret's) lie among the subproduct  $U^n$  in the domain (resp. the co-domain) of  $\llbracket \mathbf{x} \rrbracket$ . See Figure 1.



$\llbracket x \rrbracket$  with asret's  $\triangleright$  and ascoret's  $\triangleleft$ ,  
and  $\sigma_x$

$\text{Ex}(\sigma, x)$ , where  $\blacktriangleright$  (resp.  $\blacktriangleleft$ ) denotes  $\triangleright^0$  (resp.  $\triangleleft^0$ ).

Figure 1: Execution formula  $\text{Ex}(\sigma, x)$  and  $(\llbracket x \rrbracket, \sigma_x)$

**Example 4.2** Let  $x$  be in Example 4.1. Since  $\epsilon_x$  acts zero both on the unique asret  $\triangleright$  and on the unique ascoret  $\triangleleft$ ,  $\text{Ex}(\sigma, x) = \triangleleft^0 \circ (s_{U,U}^0 \otimes s_{U,U}) \circ \triangleright^0 = 0_{U^3, U^3}$ .

Finally, the execution formula is run point-wise for every enumerated set  $\nu$  in interpretation of a proof in  $\text{Rel}^{[c]}$ .

**Definition 4.7 (Execution formula  $\text{Ex}_J(\sigma, \nu)$  for  $\nu \in |\pi_{[\Delta], \Gamma}|^J$ )**

Let  $\pi_{[\Delta], \Gamma}$  be a  $\text{MALL}^{[c]}$  proof. For every  $\nu \in |\pi_{[\Delta], \Gamma}|^J$ ,  $\text{Ex}(\sigma, \nu) \in |\Gamma|^J$  is defined indexwise:

$$(\text{Ex}_J(\sigma, \nu))_j = \text{Ex}(\sigma, \nu_j) \quad \text{for every index } j \in J$$

## 4.2 Zero Convergence of Execution Formula

This subsection concerns a main proposition (Proposition 4.1), which says that communicating two proofs via mismatching pair yields zero convergence of  $\text{Ex}$ . We start with the tracing zero lemma derivable from some trace axioms.

**Lemma 4.1 (tracing zero)** For any natural number  $n \geq 1$ ,

$$\text{Tr}_{U^n, U^n}^U(0_{U^{n+1}}) = 0_{U^n} \quad (16)$$

**Proof.** First observe that  $0_{U^m} = (0_U)^m$  for any natural number  $m$  because of the uniqueness of the zero morphism. Then by superposing  $\text{Tr}_{U^{n+1}, U^{n+1}}^U(0_{U^{n+2}}) = 0_{U^n} \otimes \text{Tr}_{U, U}^U(0_{U^2})$ , thus it suffices to prove the assertion for  $n = 1$ . Second, observe the equation<sup>3</sup>

$$(0_U \otimes U) \circ s_{U, U} \circ (0_U \otimes U) = 0_U \otimes 0_U = (U \otimes 0_U) \circ s_{U, U} \circ (U \otimes 0_U) \quad (17)$$

Thus  $\text{Tr}_{U, U}^U(0_{U^2}) = 0_U \circ \text{Tr}_{U, U}^U(s_{U, U}) \circ 0_U = 0_U \circ U \circ 0_U$ , where the first eqn is by naturalities and the second eqn is by yanking.  $\square$

**Proposition 4.1 (Mismatching gives rise to zero convergence of  $\text{Ex}$ )**

For two  $\text{MALL}$  proofs  $\pi_{[\Delta_i], \Gamma_i, A_i}^i$  with  $i = 1, 2$ , let  $x_i = \delta_i \times (\gamma_i, a_i) \in |\pi_{[\Delta_i], \Gamma_i, A_i}^i|$  so that  $a_i \in |A_i|$  with  $a_1 \neq a_2$ . Then

$$\text{Ex}(\sigma_{a_1, a_2}, \text{Ex}(\sigma, x_1) \otimes \text{Ex}(\sigma, x_2)) = 0_{U^{n_1+n_2}} \quad \text{where } \sigma_{a_1, a_2} = (s_{U_{a_1}, U_{a_2}})^0.$$

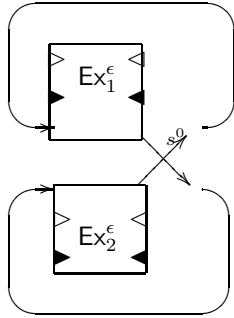
<sup>3</sup>More general setting, the natural iso  $(b \otimes U) \circ s_{U, U} \circ (a \otimes U) \cong a \otimes b \cong (U \otimes a) \circ s_{U, U} \circ (U \otimes b)$ , for any endomorphisms  $a$  and  $b$  on  $U$



Note that LHS of the assertion is, by definition, the following, in which  $\epsilon$  is the action yielded by  $\sigma_{a_1, a_2}$  (cf. Definition 4.5):

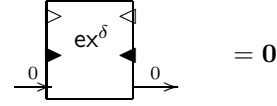
$$\begin{aligned} & \text{Tr}_{U^{n_1+n_2}, U^{n_1+n_2}}^{U^{2(1+m_1+m_2)}} ((\text{Id} \otimes \sigma_{a_1, a_2} \otimes \sigma_{\mathbf{x}_1} \otimes \sigma_{\mathbf{x}_2}) \circ (\llbracket \mathbf{x}_1 \rrbracket^{\epsilon_{\mathbf{x}_1}} \otimes \llbracket \mathbf{x}_2 \rrbracket^{\epsilon_{\mathbf{x}_1}})^\epsilon) \\ &= \text{Tr}_{U^{n_1+n_2}, U^{n_1+n_2}}^{U^{2(1+m_1+m_2)}} ((\text{Id} \otimes \sigma_{a_1, a_2}) \circ (\text{Ex}(\sigma, \mathbf{x}_1) \otimes \text{Ex}(\sigma, \mathbf{x}_2))^\epsilon) \quad \text{by nats and vanish.IIs} \end{aligned} \quad (18)$$

Thus, the L.H.S is parallel to the interpretation of the cut rule for  $a \neq a'$  in Definition 4.5. See Figure 2 (left) for the picture of the proposition.



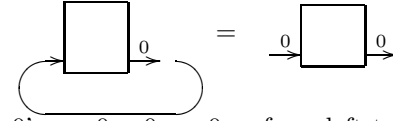
= 0

Proposition 4.1:  $\text{Ex}_i$  denotes  $\text{Ex}(\sigma, \mathbf{x}_i)$ .



= 0

Lemma 4.2 :  $\text{ex}$  denotes  $\text{ex}(\sigma, \mathbf{x})$ .



0's are  $0_U, 0_{I,U}, 0_{U,I}$  from left to right.  
the equation (19)

Figure 2: Prop 4.1 and Lem 4.2 and the eqn (19) pictorially

The proposition will be seen a direct consequence of the following Lemma:

**Lemma 4.2 (lemma for Proposition 4.1)** *For  $\mathbf{x} \in \pi_{[\Delta], \Gamma}$ , let us put  $\llbracket \mathbf{x} \rrbracket$  into the context  $(\text{Id} \otimes 0_U) \circ (-) \circ (\text{Id} \otimes 0_U)$ , allowing interaction of the asret's and the ascoret's of  $\llbracket \mathbf{x} \rrbracket$  with the two zeros  $0_U$  in the context. Zero input (resp. zero output) of asret (resp. ascoret) in this context is defined in the same manner, yielding the action, say  $\delta$ , on the asret's and the ascoret's of  $\llbracket \mathbf{x} \rrbracket$  samely as in Definition 4.5. Then,*

$$(\text{Id} \otimes 0_U) \circ \text{ex}(\sigma, \mathbf{x})^\delta \circ (\text{Id} \otimes 0_U) = 0_{U^n}$$

See Figure 2 (upper-right) depicting the equation. The lemma holds generally under identifying  $\text{ex}(\sigma, \mathbf{x})$  modulo permutations of  $U^n$ . That is,  $\text{ex}(\sigma, \mathbf{x})$  is up to the coset consisting of  $\tau^{-1} \circ \text{ex}(\sigma, \mathbf{x}) \circ \tau$  for permutations  $\tau$ , hence the assertion is independent of the choice of  $U$  for the  $0_U$ . The choice corresponds to that of one formula from  $\Gamma$ .

**Proof.** [Proof of Lemma 4.2]

Induction on the construction of  $\pi$  for  $\mathbf{x}$  in Definition 4.2. In the proof, the equation (17) in the proof of Lemma 4.1 is used. In the following, for  $i = 1, 2$ ,  $\mathbf{x}_i$  are the premises of  $\mathbf{x}$  (i.e.,  $\mathbf{y}$  and  $\mathbf{z}$  respectively  $i = 1, 2$  in Definition 4.2).  $\text{ex}_i$  denotes  $\text{ex}(\sigma, \mathbf{x}_i)$ .

(axiom)

$(\text{Id} \otimes 0_U) \circ \llbracket ax \rrbracket \circ (\text{Id} \otimes 0_U) = (\text{Id} \otimes 0_U) \circ s_{U,U} \circ (\text{Id} \otimes 0_U) = 0_U \otimes 0_U$ . The last eqn is by (17).

( $\otimes$ -rule) (case 1)  $U$  is introduced by the  $\otimes$ -rule.

$(\text{Id}_1 \otimes 0_U \otimes \text{Id}_2) \circ \triangleleft \circ (\text{ex}_1 \otimes \text{ex}_2) \circ \triangleright \circ (\text{Id}_1 \otimes 0_U \otimes \text{Id}_2)$

$= \triangleleft \circ ((\text{Id}_1 \otimes 0_U) \circ \text{ex}_1 \circ (\text{Id}_1 \otimes 0_U)) \otimes ((\text{Id}_2 \otimes 0_U) \circ \text{ex}_2 \circ (\text{Id}_2 \otimes 0_U)) \circ \triangleright = 0_{U^{n_1}} \otimes 0_{U^{n_2}}$

The last eqn is by I.H.'s on  $\llbracket \mathbf{x}_1 \rrbracket$  and  $\llbracket \mathbf{x}_2 \rrbracket$ .

( $\otimes$ -rule) (case 2) else of case 1:

In this case, we assume without loss of generality that the  $0_U$  of  $0_U \otimes \text{Id}$  lies on the domain and on the co-domain of  $\llbracket \mathbf{x}_1 \rrbracket$ . Then,  $(\text{Id} \otimes 0_U) \circ \text{ex}_1 \circ (\text{Id} \otimes 0_U) = 0_{U^{n_1}}$  by I.H. on  $\mathbf{x}_1$ . This directly implies that the co-retraction and the retraction  $(j, k)$  interpreting the  $\otimes$ -rule are acted by zero, denoted by  $(j^0, k^0)$ , w.r.t. precomposing and composing the  $0_U$  respectively since  $j$ 's output and  $k$ 's input both on  $\llbracket \mathbf{x}_1 \rrbracket$  are zeros by the above I.H. Hence, when  $(j, k)$  written by  $(\triangleleft, \triangleright)$ ,

$\triangleleft^0 \circ (0_{U^{n_1}} \otimes \mathbf{ex}_2) \circ \triangleright^0 = \triangleleft \circ (0_{U^{n_1}} \otimes (0_U \otimes \text{Id}) \circ \mathbf{ex}_2 \circ (0_U \otimes \text{Id})) \circ \triangleright = \triangleleft \circ (0_{U^{n_1}} \otimes 0_{U^{n_2}}) \circ \triangleright = 0_{U^n}$  The first eqn is by the assumption and the second eqn is by I.H. on  $\llbracket \mathbf{x}_2 \rrbracket$ .  
(cut-rule)

We assume without loss of generality that the  $0_U$  of the  $\text{Id} \otimes 0_U$  lies on the domain and on the co-domain of  $\llbracket \mathbf{x}_1 \rrbracket$ . We can write  $\mathbf{ex}(\sigma, \mathbf{x}) = \text{Tr}_{U^n, U^n}^{U^2} ((\text{Id}_1 \otimes s_{U, U} \otimes \text{Id}_2) \circ (\mathbf{ex}_1 \otimes \mathbf{ex}_2))$ , then by naturalities, LHS of the assertion is equal to

$$\begin{aligned}
& \text{Tr}_{U^n, U^n}^{U^2} ((\text{Id} \otimes 0_U) \circ ((\text{Id}_1 \otimes s_{U, U} \otimes \text{Id}_2) \circ (\mathbf{ex}_1^\delta \otimes \mathbf{ex}_2^\delta)) \circ (\text{Id} \otimes 0_U)) \\
&= \text{Tr}_{U^n, U^n}^{U^2} ((\text{Id}_1 \otimes s_{U, U} \otimes \text{Id}_2) \circ ((\text{Id} \otimes 0_U) \circ \mathbf{ex}_1^\delta \circ (\text{Id} \otimes 0_U)) \otimes \mathbf{ex}_2^\delta) \quad \text{by the assumption} \\
&= \text{Tr}_{U^n, U^n}^{U^2} ((\text{Id}_1 \otimes s_{U, U} \otimes \text{Id}_2) \circ (0_{U^{n_1+1}} \otimes \mathbf{ex}_2^\delta)) \quad \text{I.H. on } \llbracket \mathbf{x}_1 \rrbracket \\
&= \text{Tr}_{U^n, U^n}^{U^2} ((\text{Id}_1 \otimes (0_U \otimes U) \circ s_{U, U} \circ (0_U \otimes U) \otimes \text{Id}_2) \circ (0_{U^{n_1+1}} \otimes \mathbf{ex}_2^\delta)) \quad \text{dinaturality} \\
&= \text{Tr}_{U^{n_1}, U^{n_1}}^U (0_{U^{n_1+1}}) \otimes \text{Tr}_{U^{n_2}, U^{n_2}}^U ((0_U \otimes \text{Id}) \circ \mathbf{ex}_2^\delta) \quad (17) \text{ and superposing} \\
&= \text{Tr}_{U^{n_1}, U^{n_1}}^U (0_{U^{n_1+1}}) \otimes \text{Tr}_{U^{n_2}, U^{n_2}}^U ((0_U \otimes \text{Id}) \circ \mathbf{ex}_2^\delta \circ (0_U \otimes \text{Id})) \quad \text{dinaturality} \\
&= 0_{U^{n_1}} \otimes 0_{U^{n_2}} \quad \text{I.H. on } \llbracket \mathbf{x}_2 \rrbracket
\end{aligned}$$

The 1st dinaturality is via the decomposition  $0_{U^{n_1+1}} = (U^{n_1} \otimes 0_U) \circ 0_{U^{n_1}} \circ (U^{n_1} \otimes 0_U)$  and the 2nd dinaturality is via the decomposition  $0_U = 0_U \circ 0_U$ .

( $\mathfrak{A}$ -rule and additives)

Direct from the construction.

See Appendix B.2 for a pictorial proof depicting the above rewriting in each case.  $\square$

Before the proof of Proposition 4.1, let us observe a general equation derivable from some trace axioms (dinaturality and yanking), where  $f : X \otimes U \longrightarrow Y \otimes U$  and  $0_{U, I}$  (res.  $0_{I, U}$ ) is the zero morphism from  $U$  to the tensor unit  $I$  (resp. the otherway around).

$$\text{Tr}_{X, Y}^U ((\text{Id} \otimes 0_U) \circ f) = (\text{Id} \otimes 0_{U, I}) \circ f \circ (\text{Id} \otimes 0_{I, U}) \quad (19)$$

See Figure 2 (lower-right) depicting the equation.

(proof of (19))

By the decomposition  $0_U = 0_{I, U} \circ 0_{U, I}$ , LHS is  $\text{Tr}_{X \otimes I, Y \otimes I}^I ((\text{Id} \otimes 0_{U, I}) \circ f \circ (\text{Id} \otimes 0_{I, U}))$ , by dinaturality, which is equal to RHS by vanishing. (end of proof of (19))

Finally we go to;

**Proof.** [Proof of Proposition 4.1]

We prove the following instance by the above (18), where  $n = n_1 + n_2$ , since  $\sigma_{a_1, a_2} = s^0 = 0_U^2$ :

$$\text{Tr}_{U^n, U^n}^{U^2} ((\text{Id} \otimes 0_U^2) \circ (\mathbf{Ex}_1 \otimes \mathbf{Ex}_2)^\epsilon) = 0_{U^{n_1+n_2}} \quad \text{where } \mathbf{Ex}_i = \mathbf{Ex}(\sigma, \mathbf{x}_i)$$

For this, it suffices to show the following stronger equation, as  $\mathbf{ex}_i$  is  $\mathbf{Ex}_i$  ridden of the zero action on the asret's and the ascoret's:

$$\text{Tr}_{U^n, U^n}^{U^2} ((\text{Id} \otimes 0_U^2) \circ (\mathbf{ex}_1 \otimes \mathbf{ex}_2)^\epsilon) = 0_{U^{n_1+n_2}} \quad \text{where } \mathbf{ex}_i = \mathbf{ex}(\sigma, \mathbf{x}_i).$$

By superposing (after the distribution of  $\epsilon$  over  $\otimes$ ), LHS is equal to

$$\text{Tr}_{U^{n_1}, U^{n_1}}^U ((\text{Id} \otimes 0_U) \circ \mathbf{ex}_1^\epsilon) \otimes \text{Tr}_{U^{n_2}, U^{n_2}}^U ((\text{Id} \otimes 0_U) \circ \mathbf{ex}_2^\epsilon) \quad (20)$$

On the other hand, Lemma 4.2 and (19) say for all  $i = 1, 2$

$$\text{Tr}_{U^{n_i}, U^{n_i}}^U ((0_U \otimes \text{Id}) \circ \mathbf{ex}_i^\delta) = 0_{U^{n_i}}$$

Since the two actions  $\epsilon$  and  $\delta$  coincide again by (19), the equation (20) becomes equal to  $0_{U^{n_1}} \otimes 0_{U^{n_2}} = 0_{U^n}$ .  $\square$

### 4.3 Main Theorem

This section concerns a main theorem of this paper.

**Theorem 4.1 (Ex is invariant and diminishes indexes)**

Let  $\nu \in |\pi_{[\Delta], \Gamma}|^J \xrightarrow{r^b} \nu' \in |\pi'_{[\Delta'], \Gamma}|^{J'}$  be any  $\text{MALL}^{[c]}(I)$  proof transformation. Then

1.

$$\text{Ex}_J(\sigma, \nu) \upharpoonright_{J'} = \text{Ex}_{J'}(\sigma, \nu') \quad \text{and} \quad \forall j \in J \setminus J' \quad \text{Ex}(\sigma, \nu_j) = 0$$

2. In particular, when  $\pi'$  is cut-free so that  $\Delta'$  is empty, then

$$\text{Ex}_J(\sigma, \nu) \upharpoonright_{J'} = \llbracket \nu' \rrbracket \quad \text{and} \quad J' = \{j \in J \mid \text{Ex}(\sigma, \nu_j) \neq 0\}$$

**Proof.** We prove 1 accordingly to the cases of Proposition 3.1, since 2 is direct from 1 as follows: For a cut-free  $\pi'$ ,  $\sigma_{\nu'}$  is empty, hence  $\text{Ex}(\sigma, \nu_j) = \llbracket \nu'_j \rrbracket$ , where  $\llbracket \nu'_j \rrbracket \neq 0$  holds directly both from the construction of Definition 4.2 and from the non-collapsing assumption (5).

The invariance of 1 is direct by yanking axiom in case 1, and by induction on the proof  $\pi$  in cases 2 and 3: This proof method directly comes as a pointwise instance of the known method in the symmetric traced monoidal category modelling multiplicative GoI [Haghverdi Scott '06]. Thus we prove the zero convergence for the diminution of  $J$ .

(Crucial case 1)

Each instance of  $\nu$  at  $j \in J \setminus J'$  is  $\nu_j = (\delta_j, b_j, a_j, b_j, \lambda_j)$ , so that  $a_j \neq b_j$ , then  $\text{Ex}(\sigma, \nu_j) = 0$  by Proposition 4.1.

(Crucial case 2)

$J = J_1 + J_2$  diminishes into  $J_1$ . Each instance of  $\tau$  at  $j_2 \in J_2$  is  $\tau_{j_2} = (\omega, \delta_2, \gamma, (2, a_2)) \in |\&(\pi^1, \pi^2)|$ . Thus each instance of  $\nu$  at  $j_2 \in J_2$  is  $\nu_{j_2} = (\omega, \delta_2, \delta_3, (2, a_2), (1, a_1), \gamma, \xi) \in |\pi|$ . Since  $(2, a_2) \neq (1, a_1)$ , we have  $\text{Ex}(\sigma, \nu_{j_2}) = 0$  by Proposition 4.1.

(Crucial case 3)

$J$  does not diminish in this case. □

## Conclusion and Future Work

This paper offers two main contributions:

- (i) Presenting an indexed MALL system for stacking cut formulas and its relational counterpart to simulate MALL proof reduction of cut elimination.
- (ii) Constructing an execution formula for the interpretation of MALL proofs equipped with indexes. The MALL proof reduction is characterized by the convergence of the execution formula to the denotational interpretation. Furthermore, the zero convergence of the execution formula characterizes the diminution of indexes, which is peculiar to additive cut elimination.

Our explicit use of index-theoretical manipulations directly overcomes known difficulties in additive GoI. We hope that this paper, from the perspective of indexed linear logic, will shed light on an approachable understanding of the preceding literature on additive GoI, from precursory ones [Girard '95, Duchesne '09] to more recent developments [Girard '11, Seiller '16].

We discuss some future directions.

For a genuine MALL GoI without the bypass through indexed logic, a syntax-free counterpart is required to replace the indexes. We are now preparing such GoI [Hamano '15] by an algebraic ingredient of scalar extension of Girard's \*-algebra of partial isometries over a boolean polynomial ring.

In a syntactic direction, the status of Gentzen cut elimination for  $\text{MALL}(I)$  remains open since the present paper only concerns lifting the image to the indexes of MALL cut-reduction. The status will complement the reduction-free cut elimination, known derivable from the Fundamental lemma 2.1 (cf. [Bucciarelli and Ehrhard '00, Bucciarelli and Ehrhard '01, Hamano and Takemura '08]).

## References

- [Bucciarelli and Ehrhard '00] A. Bucciarelli and T. Ehrhard, On phase semantics and denotational semantics in multiplicative-additive linear logic. *Ann. Pure Appl. Logic* 102(3) 247-282 (2000)
- [Bucciarelli and Ehrhard '01] A. Bucciarelli, T. Ehrhard, On phase semantics and denotational semantics: the exponentials. *Ann. Pure Appl. Logic* 109(3): 205-241 (2001)
- [Danos and Regnier '95] V. Danos and L. Regnier, Proof-nets and the Hilbert space. In J.-Y. Girard, Y. Lafont, and L. Regnier, eds. *Advances in Linear Logic*, Cambridge University Press, (1995) 307–328
- [Duchesne '09] E. Duchesne, La Localisation en Logique : Géométrie de l'Interaction et Sémantique Dénotationnelle. Thèse de doctorat, Université Aix-Marseille II, 2009.
- [Girard '89] J-Y. Girard, Geometry of Interaction I: Interpretation of System F, in: *Logic Colloquium '88*, North-Holland, 1989, pp. 221-260.
- [Girard '95] J-Y. Girard, Geometry of Interaction III: Accommodating the Additives, in: *Advances in Linear Logic*, LNS **222**, CUP, 1995, 329–389.
- [Girard '11] J-Y. Girard, Geometry of Interaction V: Logic in the Hyperfinite Factor, *Theor. Comput. Sci.* Vol. 412 No. 20 (2011) pp. 1860-1883
- [Haghverdi Scott '06] E. Haghverdi and P. Scott, A Categorical Model for the Geometry of Interaction, *Theor. Comput. Sci.* Vol. 350 (2-3), (2006), pp. 252-274.
- [Haghverdi and Scott '11] E. Haghverdi and P. Scott, Geometry of Interaction and the Dynamics of Proof Reduction: a tutorial, in: *New Structures in Physics, Springer, Lect. Notes in Phys.*, (2011), 339-397
- [Hamano '15] M. Hamano, Geometry of Interaction for MALL via Hughes-vanGlabbeek Proof-Nets, in preparation, extended abstract 14 pgs (2015), arXiv:1503.08925
- [Hamano and Takemura '08] M. Hamano and R. Takemura, An Indexed System for Multiplicative Additive Polarized Linear Logic, *Proc. of CSL'08, LNCS*, 5213 (2008), pp. 262-277.
- [Joyal *et al.* '96] A. Joyal, R. Street, and D. Verity, Traced Monoidal Categories. *Math. Proc. Camb. Phil. Soc.* 119 (1996), pp. 447-468.
- [Seiller '16] T. Seiller, Interaction Graphs: Additives, *Ann. Pure Appl. Logic* 167 (2016), pp. 95-154.

## A Axioms of Traced Monoidal Category

**Definition A.1** (Trace axioms of the family  $\text{Tr}_{X,Y}^Z$  of (11) [Joyal *et al.* '96, Haghverdi Scott '06])

(**Natural** in  $X$ )

$$\text{Tr}_{X,Y}^Z(f) \circ g = \text{Tr}_{X',Y}^Z(f \circ (g \otimes Z)) \quad \text{where } f : X \otimes Z \longrightarrow Y \otimes Z \quad \text{and } g : X' \rightarrow X$$

(**Natural** in  $Y$ )

$$g \circ \text{Tr}_{X,Y}^Z(f) = \text{Tr}_{X',Y}^Z((g \otimes Z) \circ f) \quad \text{where } f : X \otimes Z \longrightarrow Y \otimes Z \quad \text{and } g : Y \rightarrow Y'$$

(**Dinatural** in  $Z$ )

$$\text{Tr}_{X,Y}^Z((Y \otimes g) \circ f) = \text{Tr}_{X,Y}^{Z'}(f \circ (X \otimes g)) \quad \text{where } f : X \otimes Z \longrightarrow Y \otimes Z \quad \text{and } g : Z' \rightarrow Z$$

(**Vanishing I**)

$$\text{Tr}_{X,Y}^I(f \otimes I) = f \quad \text{where } f : X \longrightarrow Y$$

(**Vanishing II**)

$$\text{Tr}_{X,Y}^{Z \otimes W}(f) = \text{Tr}_{X,Y}^Z\left(\text{Tr}_{X \otimes Z, Y \otimes Z}^W(f)\right) \quad \text{where } f : X \otimes Z \otimes W \longrightarrow Y \otimes Z \otimes W$$

(**Superposing**)

$$g \otimes \text{Tr}_{X,Y}^Z(f) = \text{Tr}_{W \otimes X, V \otimes Y}^Z(g \otimes f) \quad \text{where} \quad f : X \otimes Z \longrightarrow Y \otimes Z \quad \text{and} \quad g : W \longrightarrow V$$

(**Yanking**)

$$\text{Tr}_{X,X}^X(s_{X,X}) = X \quad \text{for the symmetry} \quad s_{X,X} : X \otimes X \longrightarrow X \otimes X$$

**Lemma A.1 (Generalized Yanking [Haghverdi and Scott '11])**

Let  $s_{Z,Y}$  denote the symmetry from  $Z \otimes Y$  to  $Y \otimes Z$ .

$$\text{Tr}_{X,Y}^Z(s_{Z,Y} \circ (f \otimes g)) = g \circ f \quad \text{where} \quad f : X \longrightarrow Z \quad \text{and} \quad g : Z \longrightarrow Y$$

**Proof.**  $LHS \stackrel{nat}{=} \text{Tr}_{Z,Y}^Z(s_{Z,Y} \circ (Z \otimes g)) \circ f \stackrel{dinat}{=} \text{Tr}_{Z,Y}^Y((Y \otimes g) \circ s_{Z,Y}) \circ f$ . Inside the trace  $(Y \otimes g) \circ s_{Z,Y} = s_{Y,Y} \circ (g \otimes Y)$ , thus  $\text{Tr}_{Z,Y}^Y(s_{Y,Y} \circ (g \otimes Y)) \stackrel{nat}{=} \text{Tr}_{Y,Y}^Y(s_{Y,Y}) \circ g \stackrel{yank}{=} g$   $\square$

## B Omitted and Pictorial Proofs

### B.1 Proof for Proposition 2.1 (Fundamental Lemma)

**Lemma B.1 ((i) implies (ii))** Let  $\nu = \pi_{[\Delta], \Gamma}$  be a proof of a sequent  $\vdash [\Delta], \Gamma$  in  $\text{MALL}^{[c]}$ . Let  $\delta \times \gamma \in |\pi_{[\Delta], \Gamma}|^J$  (for some  $J \subseteq I$ ) with  $\delta \in |\mathfrak{sl}(\Delta)|^J$  and  $\gamma \in |\Gamma|^J$ . The sequent  $\vdash_J [\Delta \langle \delta \rangle], \Gamma \langle \gamma \rangle$  has a proof  $\rho$  in  $\text{MALL}^{[c]}(I)$  such that  $\rho \upharpoonright_{\emptyset} = \pi$ .

**Proof.** By construction on the  $\text{MALL}$  proof  $\pi$ . The proof figures are referred in Definition 2.2.

(cut rule)

$\delta \times \gamma \simeq \delta_1 \times \delta_2 \times \gamma_1 \times \gamma_2$  with  $\delta_1 \times \gamma_1 \in \pi_{[\Delta_1], \Gamma_1, A}^1$  and  $\delta_2 \times \gamma_2 \in \pi_{[\Delta_2], A^\perp, \Gamma_2}^2$ . By I.H.s on  $(\delta_i \times \gamma_i)$ s, there are  $\text{MALL}(I)$ -proofs of the sequents  $\vdash_J [\Delta_1 \langle \delta_1 \rangle], \Gamma_1 \langle \gamma_1' \rangle, A \langle \gamma_1'' \rangle$  and  $\vdash_J [\Delta_2 \langle \delta_2 \rangle], A^\perp \langle \gamma_2' \rangle, \Gamma_2 \langle \gamma_2'' \rangle$  with  $\gamma_i = \gamma_i' \times \gamma_i''$ . Note that  $A \langle \gamma_1'' \rangle$  and  $A^\perp \langle \gamma_2' \rangle$  are dual formulas since they have the same domain  $J$ . Hence the cut between the dual formulas is applied to prove  $\vdash_J [\Delta_1 \langle \delta_1 \rangle, \Delta_2 \langle \delta_2 \rangle, A \langle \gamma_1' \rangle, A^\perp \langle \gamma_2' \rangle], \Gamma_1 \langle \gamma_1' \rangle, \Gamma_2 \langle \gamma_2' \rangle$ . The assertion follows since  $\Delta_1 \langle \delta_1 \rangle \times \Delta_2 \langle \delta_2 \rangle = (\Delta_1, \Delta_2) \langle \delta_1 \times \delta_2 \rangle$ .

(&-rule)

$\nu = \{(x_1, z, y, (1, a_1)) \mid (x_1, z, y, a_1) \in \nu_1\} + \{(x_2, z, y, (2, a_2)) \mid (x_2, z, y, a_2) \in \nu_2\} \simeq \nu_1 \frown \nu_2$  with  $\gamma_i \in |\pi_{[\Delta_i, \Delta], \Gamma, A_i}|^{J_i}$  and  $J = J_1 + J_2$ . By I.H.s on  $\pi^i$ s, there are  $\text{MALL}(I)$ -proofs of  $\vdash_{J_i} [\Delta_i \langle \delta_i' \rangle, \Delta \langle \delta_i'' \rangle], \Gamma \langle \gamma_i' \rangle, A_i \langle \gamma_i'' \rangle$  with  $\delta_i = \delta_i' \times \delta_i''$  and  $\gamma_i = \gamma_i' \times \gamma_i''$ . Because  $\Gamma \langle \gamma_1' \frown \gamma_2' \rangle \upharpoonright_{J_i} = \Gamma \langle \gamma_i' \rangle$  and  $\Delta \langle \delta_1'' \frown \delta_2'' \rangle \upharpoonright_{J_i} = \Delta \langle \delta_i'' \rangle$  by Lemma 2.2, the &-rule is applied to prove  $\vdash_{J_1+J_2} [\Delta_1 \langle \delta_1' \rangle, \Delta_2 \langle \delta_2' \rangle, \Delta \langle \delta_1'' \frown \delta_2'' \rangle], \Gamma \langle \gamma_1' \frown \gamma_2' \rangle, A_1 \langle \gamma_1'' \rangle \& A_2 \langle \gamma_2'' \rangle$ .  $\square$

**Lemma B.2 ((ii) implies (i))** Let  $\vdash [\Delta], \Gamma$  be a sequent of  $\text{MALL}^{[c]}$ . Let  $\nu \in (\mathfrak{sl}(\Delta) \times \Gamma)^J$  (for some  $J \subseteq I$ ) and let  $\rho$  be a proof of  $\vdash_J ([\Delta], \Gamma) \langle \nu \rangle$  in  $\text{MALL}^{[c]}(I)$ . Then  $\nu \in |(\rho \upharpoonright_{\emptyset})_{[\Delta], \Gamma}|^J$

**Proof.** By the construction on the  $\text{MALL}(I)$  proof  $\rho$ .

(cut rule)

$$\rho \text{ is } \frac{\frac{\rho^1}{\vdash_J [\Delta_1], \Gamma_1, A} \quad \frac{\rho^2}{\vdash_J [\Delta_2], A^\perp, \Gamma_2}}{\vdash_J [\Delta_1, \Delta_2, A, A^\perp], \Gamma_1, \Gamma_2} \text{ cut}$$

The conclusion is written  $\vdash_J ([\Delta_1, \Delta_2, A, A^\perp], \Gamma_1, \Gamma_2) \langle \nu \rangle$ . From the construction,  $\nu \simeq \nu_1 \times \nu_2$  so that the conclusions of  $\rho^1$  and  $\rho^2$  are respectively  $\vdash_J ([\Delta_1], \Gamma_1, A) \langle \nu_1 \rangle$  and  $\vdash_J ([\Delta_2], A^\perp, \Gamma_2) \langle \nu_2 \rangle$ . By I.H.s on  $\rho^i$ s,  $\nu_1 \in |(\rho^1 \upharpoonright_{\emptyset})_{[\Delta_1], \Gamma_1, A}|^J$  and  $\nu_2 \in |(\rho^2 \upharpoonright_{\emptyset})_{[\Delta_2], A^\perp, \Gamma_2}|^J$ . The assertion follows since  $|(\rho^1 \upharpoonright_{\emptyset})_{[\Delta_1], \Gamma_1, A}|^J \times |(\rho^2 \upharpoonright_{\emptyset})_{[\Delta_2], A^\perp, \Gamma_2}|^J \simeq |(\rho \upharpoonright_{\emptyset})_{[\Delta_1, \Delta_2, A, A^\perp], \Gamma_1, \Gamma_2}|^J$ .

(&-rule)

$$\rho \text{ is } \frac{\frac{\rho^1}{\vdash_{J_1} [\Delta_1, \Sigma], \Gamma, A_1} \quad \frac{\rho^2}{\vdash_{J_2} [\Delta_2, \Sigma], \Gamma, A_2}}{\vdash_{J_1+J_2} [\Delta_1, \Delta_2, \Sigma], \Gamma, A_1 \& A_2} \&$$

$\nu$  is of the form  $\nu_1 \frown \nu_2$  so that the conclusions of  $\rho_i$  s are  $\vdash_{J_i} ([\Delta_i, \Sigma], \Gamma, A_i) \langle \nu_i \rangle$ . By I.H.s on  $\rho^i$ s,  $\nu_i \in |(\rho^i \upharpoonright_{\emptyset})_{[\Delta_i, \Sigma], \Gamma_1, A_i}|^{J_i}$ . The assertion follows since  $|(\rho^1 \upharpoonright_{\emptyset})_{[\Delta_1, \Sigma], \Gamma, A_1}|^{J_1} \frown |(\rho^2 \upharpoonright_{\emptyset})_{[\Delta_2, \Sigma], \Gamma, A_2}|^{J_2} \simeq |(\rho \upharpoonright_{\emptyset})_{[\Delta_1, \Delta_2, \Sigma], \Gamma, A_1 \& A_2}|^{J_1 + J_2}$

□

## B.2 Pictorial Proof for Lemma 4.2

In all the cases, the upper (resp. lower) square represents  $\text{ex}_1$  (resp.  $\text{ex}_2$ ). In all the cases, each  $\mathbf{0}$  inside the dotted square arises by an induction hypothesis.

(axiom)

$$\begin{array}{c} \xrightarrow{0} \quad \diagup \quad \diagdown \quad \xrightarrow{0} = \frac{\xrightarrow{0}}{\otimes \quad \xrightarrow{0}} \end{array} \quad \begin{array}{c} \xrightarrow{0} \quad \boxed{\phantom{0}} \quad \xrightarrow{0} \\ \boxed{\phantom{0}} \end{array} = \begin{array}{c} \xrightarrow{0} \quad \boxed{\phantom{0}} \quad \xrightarrow{0} \\ \xrightarrow{0} \quad \boxed{\phantom{0}} \quad \xrightarrow{0} \end{array} = \begin{array}{c} \boxed{\mathbf{0}} \\ \boxed{\mathbf{0}} \end{array} \quad (\otimes \text{ case 1})$$

( $\otimes$  case 2)

$$\begin{array}{c} \xrightarrow{0} \quad \boxed{\phantom{0}} \quad \xrightarrow{0} \\ \triangleright \quad \triangleleft \\ \boxed{\phantom{0}} \end{array} = \begin{array}{c} \boxed{\mathbf{0}} \\ \triangleright \quad \triangleleft \\ \boxed{\phantom{0}} \end{array} = \begin{array}{c} \boxed{\mathbf{0}} \\ \xrightarrow{0} \quad \boxed{\phantom{0}} \quad \xrightarrow{0} \end{array} = \begin{array}{c} \boxed{\mathbf{0}} \\ \triangleright \quad \triangleleft \\ \boxed{\mathbf{0}} \end{array}$$

(cut)

$$\begin{array}{c} \begin{array}{c} \xrightarrow{0} \quad \boxed{\phantom{0}} \quad \xrightarrow{0} \\ \triangleright \quad \triangleleft \\ \boxed{\phantom{0}} \end{array} = \begin{array}{c} \boxed{\mathbf{0}} \\ \triangleright \quad \triangleleft \\ \boxed{\phantom{0}} \end{array} = \begin{array}{c} \boxed{\mathbf{0}} \\ \xrightarrow{0} \quad \boxed{\phantom{0}} \quad \xrightarrow{0} \end{array} \xrightarrow{\text{dinat}} \begin{array}{c} \boxed{\mathbf{0}} \\ \xrightarrow{0} \quad \boxed{\phantom{0}} \quad \xrightarrow{0} \end{array} \\ \begin{array}{c} \xrightarrow{0} \quad \boxed{\phantom{0}} \quad \xrightarrow{0} \\ \triangleright \quad \triangleleft \\ \boxed{\phantom{0}} \end{array} = \begin{array}{c} \boxed{\mathbf{0}} \\ \triangleright \quad \triangleleft \\ \boxed{\phantom{0}} \end{array} = \begin{array}{c} \boxed{\mathbf{0}} \\ \xrightarrow{0} \quad \boxed{\phantom{0}} \quad \xrightarrow{0} \end{array} \xrightarrow{\text{dinat}} \begin{array}{c} \boxed{\mathbf{0}} \\ \xrightarrow{0} \quad \boxed{\phantom{0}} \quad \xrightarrow{0} \end{array} \\ \begin{array}{c} \xrightarrow{0} \quad \boxed{\phantom{0}} \quad \xrightarrow{0} \\ \triangleright \quad \triangleleft \\ \boxed{\phantom{0}} \end{array} = \begin{array}{c} \boxed{\mathbf{0}} \\ \triangleright \quad \triangleleft \\ \boxed{\phantom{0}} \end{array} = \begin{array}{c} \boxed{\mathbf{0}} \\ \xrightarrow{0} \quad \boxed{\phantom{0}} \quad \xrightarrow{0} \end{array} = \begin{array}{c} \boxed{\mathbf{0}} \\ \otimes \\ \boxed{\mathbf{0}} \end{array} = \begin{array}{c} \boxed{\mathbf{0}} \\ \otimes \\ \boxed{\mathbf{0}} \end{array} \end{array}$$